On Proof by Infinite Descent

Alex Simpson

LFCS, School of Informatics
University of Edinburgh, UK

Background on infinitary proof theory

Finitary formal proof cannot capture arithmetic truth (Gödel)

However, infinitary formal proof does capture arithmetic truth.

The ω-rule

$$\frac{A[0] \quad A[1] \quad A[2] \quad \cdots}{A[t]}$$

proves a universally quantified statement by

$$\frac{\dfrac{A[0] \quad A[1] \quad A[2] \quad \cdots}{A[x]}}{\forall x.\, A[x]}$$

An ω-proof is an infinitely-branching well-founded tree.

A finite proof can be seen as a pattern generating an infinite proof.

For example, a finite proof by induction:

$$\dfrac{\begin{array}{cc}\Pi_0 & \Pi_s \\ A[0] & \forall x.\, A[x] \Rightarrow A[x{+}1]\end{array}}{A[t]}$$

determines an ω-proof

$$\dfrac{\begin{array}{ccccccc} & & \Pi_s & & & \Pi_s & \\ & \Pi_0 & \forall x.\, A[x] \Rightarrow A[x{+}1] & & \Pi_1 & \forall x.\, A[x] \Rightarrow A[x{+}1] & \\ \Pi_0 & A[0] & A[0] \Rightarrow A[1] & & A[1] & A[1] \Rightarrow A[2] & \\ A[0] & & A[1] & & & A[2] & \cdots \end{array}}{A[t]}$$

ω-proof: salient points

ω-proofs are a useful tool for ordinal analysis in proof theory (Schütte)

However, the important points for this talk are:

- ω-proofs are infinite proofs given by infinitely-branching well-founded trees.
- ω-provability coincides with truth.
- Finite proof by induction generates a subclass of ω-proofs.

Irrationality of $\sqrt{2}$ — informal proof by infinite descent

To prove: $\forall x, y.\, x > 0 \Rightarrow x^2 \neq 2y^2$.

Proof.

Suppose $x_0^2 = 2x_1^2$ where $x_0 > 0$

N.B., it follows that $x_0 > x_1 > 0$

Then $\exists x_2.\, x_0 = 2x_2$ since 2 is a prime factor of x_0^2 hence of x_0

So $4x_2^2 = 2x_1^2$

i.e., $x_1^2 = 2x_2^2$

By repeating argument, we produce an infinite sequence
$x_0 > x_1 > x_2 \ldots$ of numbers all > 0. #

Naïve analysis of proof

The proof applies the Principle of Infinite Descent (PID)

$$(\forall n.\,(Q(n) \Rightarrow \exists m < n.\,Q(m))) \;\Rightarrow\; \neg Q(x)$$

which (in classical logic) is equivalent to complete induction

$$(\forall n.\,(\forall m < n.\,P(m)) \Rightarrow P(n)) \;\Rightarrow\; P(x)$$

which is in turn derivable as a consequence of simple induction

$$P(0) \;\wedge\; (\forall n.\,P(n) \Rightarrow P(n+1)) \;\Rightarrow\; P(x)$$

(Conversely, complete induction trivially implies simple induction.)

Infinite proofs by infinite descent

So is proof by infinite descent just proof by induction?

An alternative take is to view the proof by infinite descent as an infinite proof, in which the full proof involves the entire construction of the infinite sequence.

What we have written down is a finite representation of this infinite proof.

There might then be other (more complex) infinite proofs by infinite descent whose underlying finite "pattern" is not given so simply as an instance of induction via PID.

Irrationality of $\sqrt{2}$ — infinite formal proof

$$\dfrac{\dfrac{}{0<x_0,\ x_0^2=2x_1^2 \Longrightarrow 0<x_1<x_0 \wedge \exists x_2.\, x_0=2x_2} \qquad \dfrac{\begin{array}{c}\vdots\\ 0<x_1,\ x_1^2=2x_2^2 \Longrightarrow\end{array}}{x_1<x_0,\ 0<x_1,\ 4x_2^2=2x_1^2 \Longrightarrow}}{0<x_0,\ x_0^2=2x_1^2 \Longrightarrow}\ (\text{Cut})$$

The infinite proof contains one infinite branch. Along this branch there occurs a sequence of terms (in fact variables)

$$x_0,\ x_1,\ x_2, \ldots$$

and an infinite sequence of left-hand-side sequent formulas

$$x_1 < x_0,\ x_2 < x_1,\ x_3 < x_2,\ \ldots$$

Totality of Ackermann's function

$$
\begin{aligned}
A(0,n) &= n+1 && \\
A(m,0) &= A(m-1,1) && m>0 \\
A(m,n) &= A(m-1,A(m,n-1)) && m,n>0
\end{aligned}
$$

Totality of Ackermann's function

$$\begin{aligned} A(m,n,r) \;\Leftrightarrow\; & (m = 0 \;\wedge\; r = n+1) \\ & \vee\; (m > 0, n = 0 \;\wedge\; A(m-1,1,r)) \\ & \vee\; (m,n > 0 \;\wedge\; \exists s.\, A(m,n-1,s) \wedge A(m-1,s,r)) \end{aligned}$$

Totality of Ackermann's function

$$\dfrac{\dfrac{}{m=0 \Longrightarrow A(m,n,n+1)} \qquad \dfrac{\dfrac{\dfrac{\vdots}{\overset{(A)}{\Longrightarrow} \exists r.A(m-1,1,r)}}{m>0, n=0 \Longrightarrow \exists r.A(m,n,r)} \qquad \dfrac{\dfrac{\vdots \qquad\qquad \vdots}{\overset{(B)}{\Longrightarrow} \exists s.A(m,n-1,s) \qquad \overset{(C)}{\Longrightarrow} \exists r.A(m-1,s,r)}}{\Longrightarrow \exists r,s.A(m,n-1,s) \wedge A(m-1,s,r)}}{m,n>0 \Longrightarrow \exists r.\, A(m,n,r)}}{\dfrac{m=0 \Longrightarrow A(m,n,n+1) \qquad m>0 \Longrightarrow \exists r.A(m,n,r)}{\Longrightarrow \exists r.A(m,n,r)}}$$

$$\begin{aligned} A(m,n,r) \Leftrightarrow\ & (m=0 \wedge r=n+1) \\ \vee\ & (m>0, n=0 \wedge A(m-1,1,r)) \\ \vee\ & (m,n>0 \wedge \exists s.\, A(m,n-1,s) \wedge A(m-1,s,r)) \end{aligned}$$

Totality of Ackermann's function

$$\dfrac{\dfrac{}{m=0 \Longrightarrow A(m,n,n+1)} \qquad \dfrac{\dfrac{\dfrac{\vdots}{\overset{(A)}{\Longrightarrow} \exists r.A(m{-}1,1,r)}}{m>0, n=0 \Longrightarrow \exists r.A(m,n,r)} \qquad \dfrac{\dfrac{\vdots}{\overset{(B)}{\Longrightarrow} \exists s.A(m,n{-}1,s)} \qquad \dfrac{\vdots}{\overset{(C)}{\Longrightarrow} \exists r.A(m{-}1,s,r)}}{\Longrightarrow \exists r,s.A(m,n{-}1,s) \wedge A(m{-}1,s,r)}}{m,n>0 \Longrightarrow \exists r.\, A(m,n,r)}}{\dfrac{m=0 \Longrightarrow A(m,n,n+1) \qquad m>0 \Longrightarrow \exists r.A(m,n,r)}{\Longrightarrow \exists r.A(m,n,r)}}$$

If an infinite path goes through (A) or (C) infinitely often, then there is an infinite sequence $m, m-1, m-2, \ldots$.

Otherwise, eventually path runs through (B) only, in which case there is an infinite sequence $n, n-1, n-2, \ldots$.

A formal system for infinite descent

First-order language for arithmetic, with terms $0, \mathsf{s}(t), t + u, t \times u$, with equality $t = u$ and strict order relation $t < u$.

Sequents $\Gamma \Longrightarrow \Delta$, with Γ, Δ finite sets of formulas.

Standard sequent calculus rules for first-order classical logic with equality, including

$$\frac{\Gamma \Longrightarrow A, \Delta \quad \Gamma, A \Longrightarrow \Delta}{\Gamma \Longrightarrow \Delta}\ (\text{Cut}) \qquad \frac{\Gamma \Longrightarrow \Delta}{\Gamma[\theta] \Longrightarrow \Delta[\theta]}\ (\text{Subst})$$

The axioms and rule from the next slide.

$$
\begin{aligned}
t < u,\, u < v &\Longrightarrow t < v \\
t < u,\, u < t &\Longrightarrow \\
&\Longrightarrow t < u,\, t = u,\, u < t \\
&\Longrightarrow 0 = t,\, 0 < t \\
t < u &\Longrightarrow \mathsf{s}(t) < \mathsf{s}(u) \\
&\Longrightarrow t < \mathsf{s}(t) \\
t < u,\, u < \mathsf{s}(t) &\Longrightarrow \\
&\Longrightarrow t + 0 = t \\
&\Longrightarrow t + \mathsf{s}(u) = \mathsf{s}(t + u) \\
&\Longrightarrow t \times 0 = 0 \\
&\Longrightarrow t \times \mathsf{s}(u) = t + (t \times u)
\end{aligned}
$$

$$
\frac{\Gamma, t = \mathsf{s}(x) \Longrightarrow \Delta}{\Gamma, 0 < t \Longrightarrow \Delta}\; x \text{ fresh}
$$

Definition of ∞-proof

An ∞-proof is a possibly infinite tree of rule applications that satisfies the following soundness condition

(SC) Along every infinite branch $(\Gamma_i \Longrightarrow \Delta_i)_i$, for some $N \geq 0$, there exists a sequence of terms $(t_i)_{i \geq N}$, the trace, such that, $\forall i \geq N$,

$$t_i = \theta_i(t_{i+1}) \text{ or } (\theta_i(t_{i+1}) = t_i) \in \Gamma_i \text{ or } (\theta_i(t_{i+1}) < t_i) \in \Gamma_i \,,$$

where, if the $(i+1)$-th rule along the path (which has conclusion $\Gamma_i \Longrightarrow \Delta_i$) is a (Subst) rule, then θ_i is the substitution used in the rule application, otherwise θ_i is the identity function.

Furthermore, $(\theta_i(t_{i+1}) < t_i) \in \Gamma_i$ holds (the trace progresses) at infinitely many i.

N.B., ∞-proofs are finitely-branching non-well-founded trees

Soundness theorem

Theorem. If a sequent $\Gamma \Longrightarrow \Delta$ has an ∞-proof then it is true.

Proof sketch. Suppose $\Gamma \Longrightarrow \Delta$ is false under some interpretation of its free variables.

The local soundness of the inference rule produces an infinite branch through the ∞-proof of $\Gamma \Longrightarrow \Delta$ such that every sequent $\Gamma_i \Longrightarrow \Delta_i$ along the branch is false under an induced interpretation ρ_i of its free variables.

Condition (SC) produces a sequence of terms $(t_i)_{i \geq N}$ along the infinite branch such that, for every $i \geq N$, either $\rho_i(t_1) = \rho_{i+1}(t_{i+1})$ or $\rho_i(t_1) > \rho_{i+1}(t_{i+1})$. Moreover, the latter holds infinitely often. #

Completeness theorem

Theorem. If $\Gamma \Longrightarrow \Delta$ is a true sequent then it has an ∞-proof without (Subst) and using only atomic applications of (Cut).

Proof idea. Simulate the sequent calculus ω-rule.

$$
\dfrac{
\dfrac{\Gamma[0] \Longrightarrow \Delta[0]}{t = 0, \Gamma[t] \Longrightarrow \Delta[t]}
\qquad
\dfrac{\dfrac{\dfrac{
\dfrac{\Gamma[s(0)] \Longrightarrow \Delta[s(0)]}{x_1 = 0, \Gamma[s(x_1)] \Longrightarrow \Delta[s(x_1)]}
\qquad
\dfrac{\dfrac{\dfrac{\Gamma[s(s(0))] \Longrightarrow \Delta[s(s(0))] \qquad \begin{matrix}\vdots\\ x_2 > 0, \Gamma[s(s(x_2))] \Longrightarrow \Delta[s(s(x_2))]\end{matrix}}{x_2 < x_1, \Gamma[s(s(x_2))] \Longrightarrow \Delta[s(s(x_2))]}}{x_1 = s(x_2), \Gamma[s(x_1)] \Longrightarrow \Delta[s(x_1)]}}{x_1 > 0, \Gamma[s(x_1)] \Longrightarrow \Delta[s(x_1)]}
}{x_1 < t, \Gamma[s(x_1)] \Longrightarrow \Delta[s(x_1)]}}{t = s(x_1), \Gamma[t] \Longrightarrow \Delta[t]}}{t > 0, \Gamma[t] \Longrightarrow \Delta[t]}
}{\Gamma[t] \Longrightarrow \Delta[t]}
$$

Digression: a proof-theoretic project

It should be possible to give a syntactic proof of the eliminability of (Subst) and non-atomic instances of (Cut) from ∞-proofs.

A promising approach is to use Mints' continuous cut elimination.

The novelty compared with other applications of continuous cut elimination would be to show that the soundness condition (SC) is preserved under cut elimination.

(In standard applications of continuous cut elimination, e.g., to ω-proofs, one instead shows that well-foundedness is preserved by cut elimination.)

Irrationality of $\sqrt{2}$ — regular infinite proof

$$\dfrac{\dfrac{}{0<x_0,\ x_0^2=2x_1^2 \Longrightarrow 0<x_1<x_0 \wedge \exists x_2.\, x_0=2x_2} \qquad \dfrac{\dfrac{\vdots}{0<x_0,\ x_0^2=2x_1^2 \overset{(*)}{\Longrightarrow}}}{0<x_1,\ x_1^2=2x_2^2 \Longrightarrow}\,(\text{Subst})}{x_1<x_0,\ 0<x_1,\ 4x_2^2=2x_1^2 \Longrightarrow}\,(\text{Cut})$$

$$0<x_0,\ x_0^2=2x_1^2 \overset{(*)}{\Longrightarrow}$$

The infinite proof contains one infinite branch. Along this branch we have the trace

$$x_0,\ x_0,\ x_1,\ x_0,\ x_0,\ x_1,\ x_0,\ x_0,\ x_1,\ \ldots$$

which progresses infinitely often via

$$x_1 < x_0,\ x_1 < x_0,\ x_1 < x_0,\ \ldots$$

Totality of Ackermann's function: regular infinite proof

$$\dfrac{\dfrac{}{m=0 \Longrightarrow A(m,n,n+1)} \qquad \dfrac{\dfrac{\vdots}{\overset{(*)}{\Longrightarrow} \exists r.A(m,n,r)}}{\Longrightarrow \exists r.A(m-1,1,r)} \qquad \dfrac{\dfrac{\vdots}{\overset{(*)}{\Longrightarrow} \exists s.A(m,n,s)}}{\Longrightarrow \exists s.A(m,n-1,s)} \qquad \dfrac{\vdots}{\overset{(*)}{\Longrightarrow} \exists r.A(m,n,r)}}{\Longrightarrow \exists r.A(m-1,s,r)}}{\dfrac{m>0, n=0 \Longrightarrow \exists r.A(m,n,r)}{\Longrightarrow \exists r,s.A(m,n-1,s) \wedge A(m-1,s,r)}} \qquad \dfrac{}{m,n>0 \Longrightarrow \exists r.\, A(m,n,r)}}{\dfrac{}{m=0 \Longrightarrow A(m,n,n+1)} \qquad \dfrac{}{m>0 \Longrightarrow \exists r.A(m,n,r)}}{\overset{(*)}{\Longrightarrow} \exists r.A(m,n,r)}$$

Regular ∞-proofs (a.k.a. circular/cyclic proofs)

An infinite tree is regular if it has only finitely many distinct subtrees.

Equivalently, a regular tree is a tree that is generated by a finite directed (cyclic) graph

A regular ∞-proof is an ∞-proof whose proof tree is regular.

The idea is that regular ∞-proofs correspond to finite proofs by infinite descent. They form an effective proof system because:

Proposition. The soundness condition (SC) is decidable over regular proof trees, presented as finite directed graphs.

Proof method. This follows from the theory of Büchi automata. One can show that (SC) is an ω-regular property.

Corollary. Regular ∞-proofs are not complete for establishing truth.

ω-proofs and ∞-proofs compared

ω-proofs are infinitely-branching well-founded trees
∞-proofs are finitely-branching non-well-founded trees

ω-provability coincides with truth coincides with ∞-provability

Proof by induction is finitary counterpart of ω-proof.
Regular ∞-proof is finitary counterpart of ∞-proof.

Main contribution of talk:

$$\text{provability by induction} \;=\; \text{regular } \infty\text{-provability}$$

More informally,

$$\text{proof by induction} \;=\; \text{regular proof by infinite descent}$$

Main theorem, formally

Theorem. A sequent $\Gamma \Longrightarrow \Delta$ has a regular ∞-proof if and only if the implication $\bigwedge \Gamma \Rightarrow \bigvee \Delta$ is provable in Peano Arithmetic (PA).

Main theorem, formally

Theorem. A sequent $\Gamma \Longrightarrow \Delta$ has a regular ∞-proof if and only if the implication $\bigwedge \Gamma \Rightarrow \bigvee \Delta$ is provable in Peano Arithmetic (PA).

Proof that PA provable implies regular ∞-provable (cf. [BS]).
Enough to give regular ∞-proof for every instance of the Induction Schema of PA.

$$\dfrac{\dfrac{\dfrac{\dfrac{A[0], \forall y.(A[y] \Rightarrow A[s(y)]) \overset{(*)}{\Longrightarrow} A[x_0]}{A[0], \forall y.(A[y] \Rightarrow A[s(y)]) \Longrightarrow A[x_1]}\,(\text{Subst})}{x_1 < x_0, A[0], \forall y.(A[y] \Rightarrow A[s(y)]) \Longrightarrow A[s(x_1)]}}{x_0 = s(x_1), A[0], \forall y.(A[y] \Rightarrow A[s(y)]) \Longrightarrow A[x_0]}}{x_0 > 0, A[0], \forall y.(A[y] \Rightarrow A[s(y)]) \Longrightarrow A[x_0]} \qquad \dfrac{}{x_0 = 0, A[0] \Longrightarrow A[x_0]}$$

$$\dfrac{x_0 = 0, A[0] \Longrightarrow A[x_0] \qquad x_0 > 0, A[0], \forall y.(A[y] \Rightarrow A[s(y)]) \Longrightarrow A[x_0]}{A[0], \forall y.(A[y] \Rightarrow A[s(y)]) \overset{(*)}{\Longrightarrow} A[x_0]}$$

Towards proof of converse

Lemma 1. If G is a finite graph representing a regular ∞-proof Π_G then ACA_0 proves "Π_G satisfies (SC)".

Proof outline. There exists a Büchi-automaton M whose emptiness witnesses that Π_G satisfies (SC). Via a formalization of Büchi's complemenation theorem, ACA_0 proves "M empty implies Π_G satisfies (SC)". And, trivially, ACA_0 proves "M empty". □

Lemma 2. For every $n \geq 0$, ACA_0 proves "if Π is an ∞-proof of $\Gamma \Longrightarrow \Delta$, containing formulas of complexity at most Σ^0_n, then, for any assignment of numbers to free variables, the formula $\bigwedge \Gamma \Rightarrow \bigvee \Delta$ is Σ^0_n-true".

Proof. The soundness proof for ∞-proofs is directly formalizable. □

(N.B., the restriction to Σ^0_n formulas is needed for the truth predicate to be expressible.)

Proof that regular ∞-provable implies PA provable.

Suppose G is a finite graph representing a regular ∞-proof Π_G with conclusion $\Gamma \Longrightarrow \Delta$.

Let n be such that every formula in G is at most Σ^0_n.

By Lemma 1, ACA_0 proves "Π_G satisfies (SC)".

By Lemma 2, ACA_0 proves "for any assignment to free variables, the formula $\bigwedge \Gamma \Rightarrow \bigvee \Delta$ is Σ^0_n-true".

By the reflection property of the Σ^0_n-truth predicate, ACA_0 proves $\bigwedge \Gamma \Rightarrow \bigvee \Delta$.

But ACA_0 is conservative over PA.

Hence, PA proves $\bigwedge \Gamma \Rightarrow \bigvee \Delta$. $\square$

Open Problem. Give a direct syntactic translation from regular ∞-proofs to proofs in PA.

The combinatorics of this seem nontrivial.

Similar syntactic translations should be able to resolve the following questions.

Conjecture, cf. [BS]. Non-atomic cuts and the (Subst) rule are not eliminable from regular ∞-proofs.

Conjecture [BS]. Regular ∞-proofs for mutual inductive definitions are conservative over proofs by induction.

Question. Are intuitionistic regular ∞-proofs conservative over Heyting Arithmetic?

Given that regular ∞-provability coincides with inductive provability, are there any benefits of using ∞-proofs as a proof method?

Some possibilities:

- The translation from inductive proofs to regular ∞-proofs involves just a linear blow-up. Perhaps the reverse translation is more expensive; i.e., perhaps regular ∞-proofs are more succinct.
- Regular ∞-proofs potentially allow the logical complexity of proofs to be reduced, see next slide.
- The methodology easily adapts to nested least/greatest fixpoints, and to settings in which (co)inductive proof is not viable since relevant (co)induction hypotheses not expressible; see slide 30.
- Other wider classes of ∞-proofs than the regular ones might provide (more powerful?) effective proof systems: e.g., pushdown proofs, proofs generated by (higher-order) recursion schemes.

Totality of Ackermann's function

$$\dfrac{\dfrac{}{m=0 \Longrightarrow A(m,n,n+1)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Longrightarrow \exists r.A(m,n,r)}(*)}{\Longrightarrow \exists r.A(m-1,1,r)} \qquad \dfrac{\dfrac{\dfrac{}{\Longrightarrow \exists s.A(m,n,s)}(*)}{\Longrightarrow \exists s.A(m,n-1,s)} \qquad \dfrac{\dfrac{}{\Longrightarrow \exists r.A(m,n,r)}(*)}{\Longrightarrow \exists r.A(m-1,s,r)}}{\Longrightarrow \exists r,s.A(m,n-1,s) \wedge A(m-1,s,r)}}{m,n>0 \Longrightarrow \exists r.\, A(m,n,r)}}{m>0 \Longrightarrow \exists r.A(m,n,r)}}{\Longrightarrow \exists r.A(m,n,r)}(*)$$

The regular ∞-proof involves only Σ^0_1 formulas.
Ackermann's function is not provably total using Σ^0_1 induction.
So Σ^0_1 infinite descent is not conservative over Σ^0_1 induction.

Question. What logical theory is given by Σ^0_1 regular ∞-proofs?

Potential application: process verification

Sequents $\Gamma \Longrightarrow \Delta$ where Γ, Δ finite sets of assertions of the form $p : A$, stating that process p (algebra) satisfies property A (coalgebra).

Example uses interleaving parallel $x|y$ and property $U =_{\text{def}} \mu X.\,[a]X$.

$$\dfrac{\dfrac{\dfrac{x:U,\ y:U \overset{(*)}{\Longrightarrow} x|y:U}{x':U,\ y:U \Longrightarrow x'|y:U}\,(\text{Subst})}{x:[a]U,\ y:U,\ x \xrightarrow{a} x' \Longrightarrow x'|y:U} \qquad \dfrac{\dfrac{x:U,\ y:U \overset{(*)}{\Longrightarrow} x|y:U}{x:U,\ y':U \Longrightarrow x|y':U}\,(\text{Subst})}{x:U,\ y:[a]U,\ y \xrightarrow{a} y' \Longrightarrow x|y':U}}{x:U,\ y:U,\ x \xrightarrow{a} x' \Longrightarrow x'|y:U \qquad x:U,\ y:U,\ y \xrightarrow{a} y' \Longrightarrow x|y':U}$$

$$\dfrac{x:U,\ y:U,\ x|y \xrightarrow{a} z \Longrightarrow z:U}{x:U,\ y:U \Longrightarrow x|y:[a]U}$$

$$x:U,\ y:U \overset{(*)}{\Longrightarrow} x|y:U$$

Recap of questions and problems

1. Give syntactic proof of non-atomic cut-elimination for ∞-proofs.
2. Give direct translation from regular ∞-proofs to proofs in PA.
3. Quantify blow-up required when translating a regular ∞-proof to a PA proof.
4. Show non-elimininability of non-atomic cuts from regular ∞-proofs.
5. Prove equivalence of regular ∞-proof and proof by induction for mutual inductive definitions. (Main conjecture of [BS])
6. Are intuitionistic regular ∞-proofs conservative over HA?
7. Characterise logical theory given by Σ^0_1 regular proofs.
8. Investigate wider classes of ∞-proofs with finite presentations; e.g., pushdown, generated by (higher-order) recursion schemes.
9. Find convincing applications.

Main reference plus selected related work (chronological)

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