# Optimal Tableau Algorithms for Coalgebraic Logics

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# The decision problem

**Satisfiability wrt a set of global assumptions ("TBox")**

Given a formula A and a set of formulas $\Delta$ (global assumptions) of some coalgebraic modal logic.
Is A satisfiable in some state of a model $\mathcal{M}$ with $\mathcal{M} \models \Delta$?

# Example: London Underground

## Global assumptions

$$\begin{aligned} \texttt{TubeStrike} &\sqsubseteq \texttt{PayRise} \sqcup \exists \texttt{tomorrow}.\texttt{TubeStrike} \\ \texttt{PayRise} &\sqsubseteq \exists \texttt{tomorrow}.\texttt{Redundancies} \\ \texttt{Redundancies} &\sqsubseteq \exists \texttt{tomorrow}.\texttt{TubeStrike} \\ \texttt{RoyalWedding} &\sqsubseteq \texttt{PayRise} \end{aligned}$$

## Questions

Is $\exists \texttt{tomorrow}.(\neg \texttt{TubeStrike}) \sqcap \texttt{RoyalWedding}$ satisfiable?
Is $\exists^2 \texttt{tomorrow}.(\neg \texttt{TubeStrike}) \sqcap \texttt{RoyalWedding}$ satisfiable?

# Decision procedures

**Tableaux**

| | |
|---|---|
| advantage | intuitive, relatively easy to implement |
| problematic | worst case complexity (often NExpTime) |

**Automata**

| | |
|---|---|
| advantage | yields optimal upper bounds for complexity |
| problematic | average case = worst case complexity |

# In my talk

## Results

- ExpTime decidability for a family of modal logics with global reasoning.
- A simple tableau algorithm using global caching.
- Tableau algorithm can in fact be viewed as non-emptiness test of certain automata.

# Family of modal logics

## Examples

- basic modal logic ($\Box A \equiv$ "in all successors A holds" )
- graded modal logic ($\langle n \rangle A \equiv$ "in at least n successors A holds ")
- probabilistic modal logic ($\langle p \rangle A \equiv$ "A holds with probability at least p in the next state")
- combinations!

## Unifying framework

Coalgebraic modal logic.

# Example: Basic Modal Logic

## Possible World Semantics of standard modal logic

$$X \xrightarrow{\gamma} \mathcal{P}X \text{ to interpret } \Box A \text{ as "necessarily } A\text{"}$$

$$x \models \Box A \iff \boxed{\forall x' \in \gamma(x) : x' \in [\![A]\!]}$$

Define "semantics" of a modality as predicate lifting:

$$\begin{aligned} [\![\Box]\!] &: \mathcal{P}X \to \mathcal{P}\mathcal{P}X \\ [\![\Box]\!](U) &:= \{V \subseteq X \mid V \subseteq U\} \end{aligned}$$

Rephrase semantics:

$$\boxed{x \models \Box A \iff \gamma(x) \in [\![\Box]\!]([\![A]\!])}$$

# Coalgebraic modal logic: abstractly

## Syntax

Propositional logic + n-ary modalities:

$$A ::= p \mid \overline{p} \mid A_1 \wedge A_2 \mid A_1 \vee A_2 \mid \heartsuit(A_1, \cdots, A_n), \heartsuit \in \Lambda,$$

where $\Lambda$ is a set of modalities.

## Semantics

Formulas are interpreted over T-models:

$$X \xrightarrow{\langle \pi, \gamma \rangle} \mathcal{P}(PV) \times TX$$

- X is a set (of "states"),
- T is a set functor (the "transition type").

# Interpretation of formulas

## Λ-structure

A functor T together with "predicate liftings"

$$[\![\heartsuit]\!]_X : \mathcal{P}X \to \mathcal{P}TX$$

for each $\heartsuit \in \Lambda$.

## Interpretation of a modal formula

$$x \models \heartsuit(A_1, \cdots, A_n) \text{ if } \gamma(x) \in [\![\heartsuit]\!]([\![A_1]\!], \cdots, [\![A_n]\!])$$

# More Examples

- Combinations of the above:

$$\langle p \rangle \Box \, \texttt{legitimate}$$

"with probability at least p are all children of (Henry VIII) legitimate"

- coalition logic:

$$[C]A$$

"coalition C (of players) has a strategy to achieve A"

# Decidability via Tableaux

## What are tableaux?

- trees labeled with "sequents", ie., by finite sets of formulas
- generated from rules $\Gamma/\Gamma_1 \cdots \Gamma_n$ to be read "if $\Gamma$ is satisfiable, then at least one of the $\Gamma_i$'s is satisfiable"
- a tableau is closed if every branch ends on a node labeled by a propositionally inconsistent set

## Notation

We denote by $S(\Lambda)$ the collection of all sequents.

## Connection to satisfiability

A finite set $\Gamma$ is unsatisfiable if there exists a closed tableau whose root is labelled by $\Gamma$.

# One-step tableau rules

$$V = \text{“set of variables”}$$
$$\Lambda(V) = \{\heartsuit(x_1, \cdots, x_n) \mid \heartsuit \in \Lambda, x_1, \cdots, x_n \in V\}$$

**One-step rule (the “modal rules”)**

Suppose $\Lambda$ is a modal similarity type. A one-step tableau rule over $\Lambda$ is of the form

$$\frac{\Gamma_0}{\Gamma_1 \quad \dots \quad \Gamma_n}$$

where $\Gamma_0 \subseteq \Lambda(V)$ and $\Gamma_1, \dots, \Gamma_n \subseteq V$, every propositional variable occurs at most once in $\Gamma_0$.

# One-step completeness

## One-step semantics

For a valuation $\tau : \mathrm{V} \to \mathcal{P}(\mathrm{X})$ define

$$[\mathrm{x_i}]_{\mathrm{X},\tau} := \tau(\mathrm{x_i}) \subseteq \mathrm{X} \quad \text{and} \quad [\heartsuit \mathrm{x_i}]_{\mathrm{TX},\tau} := [\![\heartsuit]\!]([\mathrm{x_i}]) \subseteq \mathrm{TX}.$$

## One-step tableau completeness

A set R of one-step rules is one-step tableau sound and tableau complete if for all $\Gamma \subseteq \Lambda(\mathrm{V})$ and for all $\tau : \mathrm{V} \to \mathcal{P}(\mathrm{X})$:

$$[\Gamma]_{\mathrm{TX},\tau} \neq \emptyset \quad \Leftrightarrow \quad \begin{array}{l} \text{for all } \Gamma_0/\Gamma_1, \ldots, \Gamma_n \in \mathrm{R} \text{ and} \\ \text{for all } \sigma : \mathrm{V} \to \mathrm{V} \text{ with } \Gamma_0\sigma \subseteq \Gamma \\ \text{there exists } 1 \leq \mathrm{i} \leq \mathrm{n} \text{ s.t. } [\Gamma_\mathrm{i}\sigma]_{\mathrm{X},\tau} \neq \emptyset. \end{array}$$

# Examples

1. For the modal $\mu$-calculus, we have the rule set consisting the instances for $n \geq 0$ of

$$\frac{\Diamond p_0, \Box p_1, \ldots, \Box p_n}{p_0, p_1, \ldots, p_n}$$

2. For Coalition Logic over a set N of agents, we have

$$(C_1)\frac{[C_1]p_1, \cdots, [C_n]p_n}{p_1, \cdots, p_n}$$

$$(C_2)\frac{[C_1]p_1, \cdots, [C_n]p_n, \overline{[D]}q, \overline{[N]}r_1, \cdots, \overline{[N]}r_m}{p_1, \cdots, p_n, q, r_1, \cdots, r_m}$$

where $m, n \geq 0$. Both rules are subject to the side condition that the $C_i$ are pairwise disjoint and additionally $C_1, \ldots, C_n \subseteq D$ for $(C_2)$.

# The induced tableau system with global assumptions

The skeletal system over R with global assumptions $\Delta$ is the tableau system $(\mathrm{S}(\Lambda), \mathrm{S}(\mathrm{R}))$ where $\mathrm{S}(\mathrm{R})$ contains

$$\frac{\Gamma}{\mathrm{sat}(\Gamma)}$$

$$\frac{\Gamma_0\sigma, \Gamma'}{\Gamma_1\sigma, \Delta \quad \ldots \quad \Gamma_n\sigma, \Delta}$$

where $\Gamma \in \mathrm{S}(\Lambda)$, $\Gamma_0/\Gamma_1 \ldots \Gamma_n \in \mathrm{R}$, $\sigma : \mathrm{V} \to \mathcal{F}(\Lambda)$ is an injective substitution and $\Gamma' \in \mathrm{S}(\Lambda)$ is arbitrary.

## $\mathrm{sat}(\Gamma)$-Example

- $\mathrm{sat}(\Gamma) \subseteq \mathcal{P}(\mathrm{S}(\Lambda))$,
- $\{A_1, B, C\} \in \mathrm{sat}(\{A_1 \vee A_2, B \wedge C\})$
- $\mathrm{sat}(\Gamma) = \emptyset$ if $\Gamma$ propositionally inconsistent

# Tableau issues

## Good

- intuitive
- relatively easy to implement

## Problem

- tableau-based decision methods often do not meet the optimal complexity bound
- reason: depth-first search for an "open" tableau branch - one sequent can occur on several branches

## Solution

Global caching: the sequents are arranged in a graph; a sequent occurs at most once - and is expanded at most once!

# Abstract tableaux

## Tableau system

A tableau system is a pair $(S, R)$ where

- S is a set (of sequents) and
- R is a set of rules of the form $\Gamma/\Psi$

where $\Gamma \in S$ and $\Psi \subseteq S$ is finite.

## Closed Tableau

A sequent $\Gamma \in S$ has a closed tableau iff $\Gamma$ is an element of the least fixpoint of

$$\begin{aligned} M : \mathcal{P}(S) &\to \mathcal{P}(S), \\ M(X) &= \{\Gamma \in S \mid \exists \Psi \subseteq X. \quad \Gamma/\Psi \in R\}. \end{aligned}$$

# The game perspective

## Specification of the game

- graph game (played on a directed graph)
- 2 player: $\forall$ and $\exists$ (Abelard and Eloise)
- $\exists$ wins a play if $\forall$ can't move or if the play is infinite

## The Moves in one "round" of a play

$$\begin{array}{rcccl} S \ni & \Gamma & \longrightarrow_\forall & \Gamma/\Psi & \in R \\ R \ni & \Gamma/\Psi & \longrightarrow_\exists & \Gamma' & \in \Psi \end{array}$$

## Theorem

$\exists$ has a winning strategy at position $\Gamma \in S$ iff $\Gamma$ is satisfiable.

# Our algorithm

**Idea**

Compute the game board until you are able to decide whether $\forall$ or $\exists$ has winning strategy for a certain position.

**Desirable conditions of the "partial board"**

1. no "unexpanded" nodes: if $\Gamma$ is on the board then so are all sequents $\Gamma'$ that are reachable in one round of a play
2. a node for which we can decide whether it is a winning position for $\forall$ or $\exists$ is marked by A or E, respectively.

# Algorithm: Expansion step

Define a relation $\to_E$ ("expand") on triples $(A, U, E)$ of sets of sequents by

$$(A, U, E) \to_E (A', U', E') \quad \text{iff}$$

- $U' = (U \cup \Psi) \setminus (A \cup E)$ for some $\Gamma \in U$ with $\Gamma/\Psi \in R$,
- $A' = A$,
- $E' = E$.

Write $(A, U, E) \circlearrowleft$ if $(A, U, E) \to_E (A', U', E')$ implies that $(A, U, E) = (A', U', E')$.

Addition of missing rule conclusions to U.

# Algorithm: Propagation step

Define a relation $\rightarrow_{\mathrm{P}}$ ("propagate") on triples $(\mathrm{A}, \mathrm{U}, \mathrm{E})$ of sets of sequents by

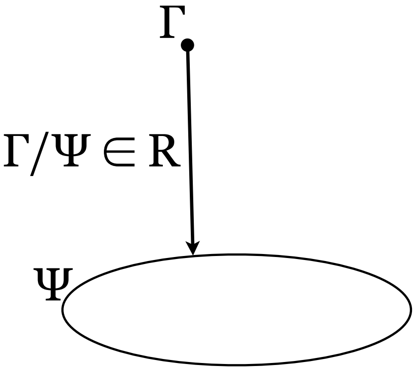
$$(\mathrm{A}, \mathrm{U}, \mathrm{E}) \rightarrow_{\mathrm{P}} (\mathrm{A}', \mathrm{U}', \mathrm{E}') \text{ iff } \mathrm{A}' = \mathrm{A} \cup \mu \mathrm{M} \quad \mathrm{E}' = \mathrm{E} \cup \nu \mathrm{W}$$

and $\mathrm{U}' = \mathrm{U}$ where $\mathrm{M}, \mathrm{W} : \mathcal{P}(\mathrm{U}) \rightarrow \mathcal{P}(\mathrm{U})$ are given by

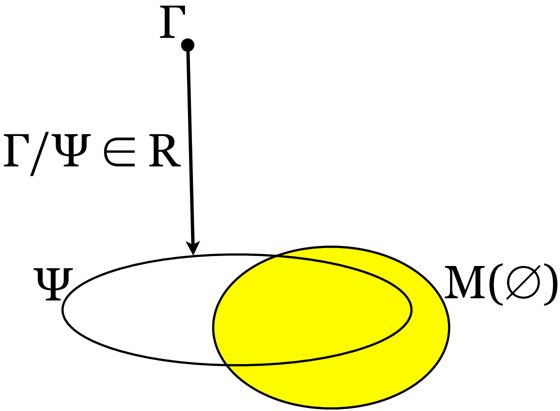
$$\mathrm{M}(\mathrm{X}) = \{\Gamma \in \mathrm{U} \mid \exists \Psi \subseteq \mathrm{X} \cup \mathrm{A} \, (\Gamma/\Psi \in \mathrm{R})\}$$
$$\mathrm{W}(\mathrm{X}) = \{\Gamma \in \mathrm{U} \mid \forall \Gamma/\Psi \in \mathrm{R} \, (\Psi \cap (\mathrm{X} \cup \mathrm{E})) \neq \emptyset\}$$

for propagation.

# Propagation: Update of A



# Propagation: Update of A



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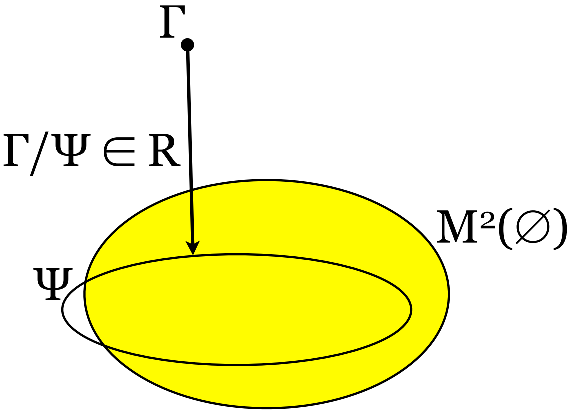
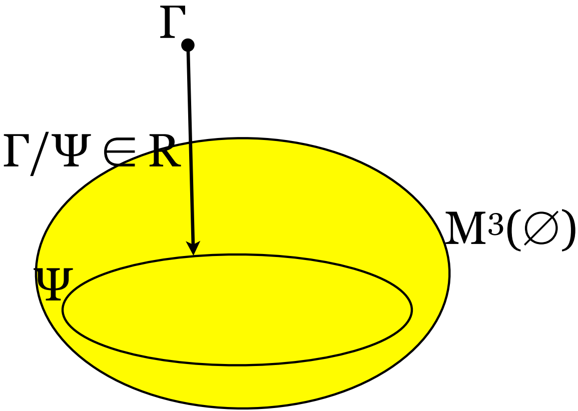
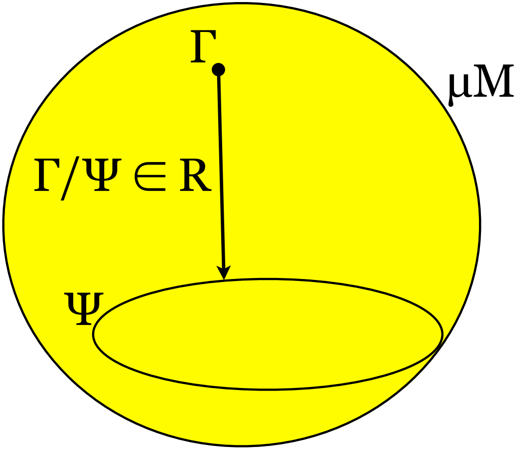


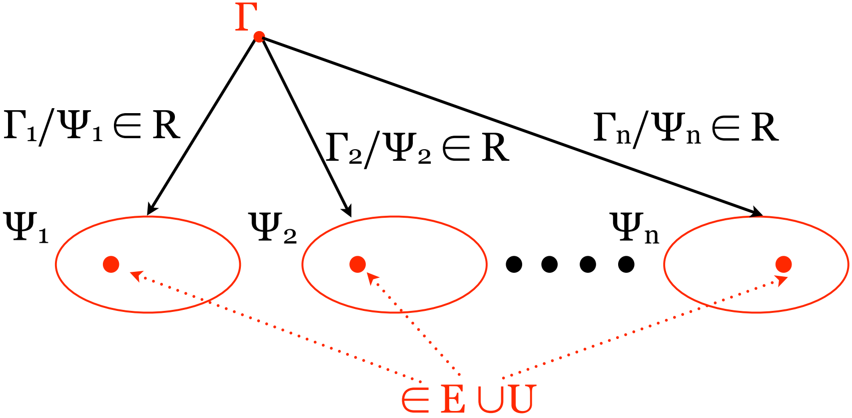
# Propagation: Update of A



# Propagation: Update of A



# Propagation: Update of E



# The algorithm in detail (simplified)

Decide whether $\Gamma \in \mathrm{S}$ has a closed tableau.

1. Let $(\mathrm{A}, \mathrm{U}, \mathrm{E}) = (\emptyset, \{\Gamma\}, \emptyset)$
2. while $(\mathrm{A}, \mathrm{U}, \mathrm{E}) \not\emptyset$ do
   - 2.1 let $(\mathrm{A}, \mathrm{U}, \mathrm{E}) = (\mathrm{A}', \mathrm{U}', \mathrm{E}')$ where $(\mathrm{A}, \mathrm{U}, \mathrm{E}) \rightarrow_{\mathrm{E}} (\mathrm{A}', \mathrm{U}', \mathrm{E}')$
   - 2.2 (opt.) let $(\mathrm{A}, \mathrm{U}, \mathrm{E}) = (\mathrm{A}', \mathrm{U}', \mathrm{E}')$ where $(\mathrm{A}, \mathrm{U}, \mathrm{E}) \rightarrow_{\mathrm{P}} (\mathrm{A}', \mathrm{U}', \mathrm{E}')$
3. let $(\mathrm{A}, \mathrm{U}, \mathrm{E}) = (\mathrm{A}', \mathrm{U}', \mathrm{E}')$ where $(\mathrm{A}, \mathrm{U}, \mathrm{E}) \rightarrow_{\mathrm{P}} (\mathrm{A}', \mathrm{U}', \mathrm{E}')$
4. say “yes” if $\Gamma \in \mathrm{A}$ and “no” otherwise

# Decidability

**Theorem**

Suppose that $\Lambda$ is a modal similarity type and T is a $\Lambda$-structure. If R is a one-step tableau sound and complete set of one-step rules that admits contraction, then the tableau algorithm decides satisfiability of $\Gamma$ in Mod($\Delta$) in at most exponential time (w.r.t. size($\Gamma, \Delta$)), if R is exponentially tractable.

# Connection to automata

Simplifcation:

- assume that all modalities are monotone

# Looping automata

**Definition**

A looping automaton is a triple $(Q, \delta, q_I)$ where

- $Q$ is a finite set of states
- $\delta : Q \rightarrow \mathcal{P}\mathcal{P}(\Lambda(Q) \cup PV \cup \overline{PV})$ is a transition function
- $q_I \in Q$ is the initial state

Simplified version of the automata for the coalgebraic $\mu$-calculus proposed by Fontaine/Leal/Venema at ICALP 2010.

# Acceptance

A pointed T-model is a triple $(X, \langle \pi, \gamma \rangle, x_I)$ is a triple where

- $(X, \langle \pi, \gamma \rangle)$ is a T-model and
- $x_I \in X$ is a designated point/state.

An automaton $(Q, \delta, q_I)$ accepts or rejects pointed T-model $(X, \langle \pi, \gamma \rangle, x_I)$.

Think of automata as formulas!

# Acceptance Game

An automaton $(Q, \delta, q_I)$ accepts $(X, \langle \pi, \gamma \rangle, x_I)$ if $\exists$ has a winning strategy at position $(x_I, q_I)$ in the following 2-player graph game:

$$
\begin{array}{lll}
(x, q) \in X \times Q & \longrightarrow_{\exists} & \tau : Q \to \mathcal{P}(X) \\
 & & \quad \text{with } \; x \in [\Gamma]_{X,\pi} \\
 & & \quad \text{and } \; \gamma(x) \in [\Gamma]_{TX,\tau} \\
 & & \quad \text{for some } \Gamma \in \delta(q) \\
 & & \\
 & \longrightarrow_{\forall} & (x', q') \in X \times Q \text{ s.t. } \; x' \in \tau(q')
\end{array}
$$

# Connection tableaux/automata: Formula automaton

Let $\Gamma \in \mathrm{S}(\Lambda)$ be a sequent and let $\Delta \in \mathrm{S}(\Lambda)$ be a sequent corresponding to the set of global assumptions.

Define a looping automaton $\mathbb{A}_{\Gamma}^{\Delta} = (\mathrm{Q}, \delta, \mathrm{q_I})$ by putting

- $\mathrm{Q} := \mathrm{S}(\Gamma, \Delta)$,
- $\delta(\Sigma) := \mathrm{sat}(\Sigma, \Delta)$
- $\mathrm{q_I} := \Gamma$.

# Non-emptiness game using one-step rules

The non-emptiness game $\mathcal{G}_{\neq\emptyset}(\mathbb{A}^{\Delta}_{\Gamma})$ is (equivalent to) the following 2-player graph game:

$$
\begin{array}{lll}
\Sigma \in \mathrm{S}(\Gamma, \Delta) & \longrightarrow_{\exists} & \Gamma' \in \delta(\Sigma) \\
 & \longrightarrow_{\forall} & \left(\frac{\Gamma_0}{\Gamma_1 \cdots \Gamma_n}, \sigma\right) \quad \text{with} \quad \frac{\Gamma_0}{\Gamma_1 \cdots \Gamma_n} \in \mathrm{R} \text{ and} \\
 & & \qquad \sigma : \mathrm{V} \to \mathrm{Q} \text{ such that } \Gamma_0\sigma \subseteq \Gamma' \\
 & & \\
 & \longrightarrow_{\exists} & \Gamma_i\sigma \in \mathrm{Q} \quad \text{with } 1 \leq i \leq n
\end{array}
$$

Game board exponential in the size of the automaton if we assume that set of one-step rules is exponentially tractable.

# Conclusions

- Algorithm meets optimal complexity bounds
- non-determinisitc, but one execution suffices ($\Rightarrow$ room for heuristics
- global caching for coalgebraic description logics (IJCAR 2010)
- global caching for coalgebraic fixpoint logics: yes, but need to add extra structure, eg. states of a parity automaton
- Clarify expressivity of looping automata.
- Connections with work by Baader et al. on automata vs. tableaux.
- Explore close connection with Voronkov's "inverse method".