MV-algebras and fuzzy topologies: Stone duality extended

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Outline

1. MV-algebras and their reducts
2. Semisimple and hyperarchimedean MV-algebras
3. MV-topologies
4. Stone MV-spaces and semisimple MV-algebras

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MV-algebras

Definition

An MV-algebra $\langle A, \oplus, ^*, 0\rangle$ is an algebra of type (2,1,0) such that

- $\langle A, \oplus, 0\rangle$ is a commutative monoid,
- $(x^*)^* = x$,
- $x \oplus 0^* = 0^*$,
- $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

The MV-algebra $[0, 1]$

$\langle [0, 1], \oplus, ^*, 0\rangle$, with $x \oplus y := \min\{x + y, 1\}$ and $x^* := 1 - x$, is an MV-algebra, called standard. It generates the variety of MV-algebras both as a variety and as a quasi-variety.

Further operations and properties

Operations

- $x \leq y$ if and only if $x^* \oplus y = 1$,
- $1 = 0^*$,
- $x \odot y = (x^* \oplus y^*)^*$,
- $\leq$ defines a structure of bounded lattice.

Properties

- $\oplus$, $\odot$ and $\wedge$ distribute over any existing join.
- $\oplus$, $\odot$ and $\vee$ distribute over any existing meet.
- De Morgan laws hold both for weak and strong conjunction and disjunction:
  - $x \wedge y = (x^* \vee y^*)^*$ and $x \vee y = (x^* \wedge y^*)^*$,
  - $x \odot y = (x^* \oplus y^*)^*$ and $x \oplus y = (x^* \odot y^*)^*$.

MV and Boolean algebras

$\mathcal{B}$oole $\subseteq \mathcal{MV}$

Boolean algebras form a subvariety of the variety of MV-algebras. They are the MV-algebras satisfying the equation $x \oplus x = x$.

The Boolean center

Let A be an MV-algebra.

- $a \in A$ is called idempotent or Boolean if $a \oplus a = a$.
- $a \oplus a = a$ iff $a \odot a = a$.
- a is Boolean iff a^* is.
- $\mathrm{B}(A) = \{a \in A \mid a \oplus a = a\}$ is a Boolean algebra, called the Boolean center of A. It is, in fact, the largest Boolean subalgebra of A.

Semirings and quantales

Definition

A semiring is a structure $\langle S, +, \cdot, 0 \rangle$ such that

- $\langle S, +, 0 \rangle$ is a commutative monoid,
- $\langle S, \cdot \rangle$ is a semigroup,
- $\cdot$ distributes over $+$ from either side.

Definition

A quantale $\langle Q, \bigvee, \cdot, \bot \rangle$ is a sup-lattice equipped with a monoid operation $\cdot$ which distributes over arbitrary joins.

Reducts of MV-algebras

[Di Nola–Gerla B., 2005]

For any MV-algebra A, $\langle A, \vee, \odot, 0, 1\rangle$ and $\langle A, \wedge, \oplus, 1, 0\rangle$ are (commutative, unital, additively idempotent) semirings, isomorphic under the negation.

So, if A is complete, $\langle A, \bigvee, \odot, 0, 1\rangle$ and $\langle A, \bigwedge, \oplus, 1, 0\rangle$ are isomorphic (commutative, unital) quantales.

Moreover, also $\langle A, \vee, \oplus, 0\rangle$ and $\langle A, \wedge, \odot, 1\rangle$ are isomorphic semirings and, if A is complete, $\langle A, \bigvee, \oplus, 0\rangle$ and $\langle A, \bigwedge, \odot, 1\rangle$ are isomorphic quantales.

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Ideals and congruences in $\mathcal{MV}$

Definition

A subset I of an MV-algebra A is called an ideal if

- $0 \in I$;
- I is downward closed;
- $a \oplus b \in I$ for all $a, b \in I$.

Proposition

Let A be an MV-algebra. For any MV-algebra congruence $\sim$ on A, $[0]_\sim$ is an ideal of A. Conversely, for any ideal I, the relation $\sim_I$ defined by "$a \sim_I b$ iff $d(a, b) := (a \odot b^*) \oplus (b \odot a^*) \in I$" is the unique congruence on A whose zero-class is I.

Maximal ideals

Max A

The set of all maximal ideals of A is denoted by $\operatorname{Max} A$.
The radical of A is defined as the intersection of all maximal ideals: $\operatorname{Rad} A := \bigcap \operatorname{Max} A$.

Proposition

If M is a proper ideal of A then the following are equivalent:

- *M is maximal;*
- *for any $a \in A$, if $a \notin M$ then there exists $n \in \omega$ such that $(a^*)^n \in M$.*

Semisimple algebras

Definition (from Universal Algebra)

An algebra A is called semisimple if it is subdirect product of simple algebras.

Proposition

An MV-algebra A is semisimple if and only if $\mathrm{Rad}\, A = \{0\}$.

$\mathcal{MV}^{\mathrm{ss}}$

The class of semisimple MV-algebras form a full subcategory of $\mathcal{MV}$ that we shall denote by $\mathcal{MV}^{\mathrm{ss}}$.
It is worth noticing that, although $\mathcal{MV}^{\mathrm{ss}}$ is NOT a variety (it is closed under $\mathbb{S}$ and $\mathbb{P}$, but not under $\mathbb{H}$), it contains $[0,1]$, $\mathcal{B}$oole, and free, projective, σ-complete and complete MV-algebras.

Semisimple MV-algebras are algebras of fuzzy sets

Theorem [Belluce, 1986]

A is isomorphic to a subalgebra of $[0,1]^{\operatorname{Max} A}$, for any $A \in \mathcal{MV}^{ss}$.

Sketch of the proof.

- For any $M \in \operatorname{Max} A$, A/M is simple.
- [Chang, 1959]: Any simple MV-algebra is an archimedean chain, hence it is isomorphic to a (unique) subalgebra of $[0,1]$.
- So there exists a unique embedding $\iota_M : A/M \longrightarrow [0,1]$.
- Let $\varphi_M : A \longrightarrow A/M$ be the natural projection.
- $\forall a \in A$, let $\hat{a} : M \in \operatorname{Max} A \longmapsto \iota_M(\varphi_M(a)) \in [0,1]$.
- The map $\iota : a \in A \longmapsto \hat{a} \in [0,1]^{\operatorname{Max} A}$ is an MV-algebra embedding.

Hyperarchimedean algebras

Definition

Let A be an MV-algebra. An element $a \in A$ is archimedean if it satisfies the following equivalent conditions:

1. there exists a positive integer n such that $na \in \mathrm{B}(A)$;
2. there exists a positive integer n such that $a^* \vee na = 1$;
3. there exists a positive integer n such that $na = (n+1)a$.

Definition

An MV-algebra A is called hyperarchimedean if all of its elements are archimedean.

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Open sets

$\langle X, \Omega \rangle$ topological space

$\langle \{0,1\}^X, \bigvee, \bigwedge, ^*, \mathbf{0}, \mathbf{1} \rangle$ is a complete Boolean algebra.

- $\langle \Omega, \bigvee, \mathbf{0} \rangle$ is a sup-sublattice of $\langle \{0,1\}^X, \bigvee, \mathbf{0} \rangle$,
- $\langle \Omega, \wedge, \mathbf{1} \rangle$ is a meet-subsemilattice of $\langle \{0,1\}^X, \wedge, \mathbf{1} \rangle$.

$\langle X, \Omega \rangle$ MV-topological space

$\langle [0,1]^X, \bigvee, \bigwedge, \oplus, \odot, ^*, \mathbf{0}, \mathbf{1} \rangle$ is a complete MV-algebra.

- $\langle \Omega, \bigvee, \oplus, \mathbf{0} \rangle$ is a subquantale of $\langle [0,1]^X, \bigvee, \oplus, \mathbf{0} \rangle$,
- $\langle \Omega, \wedge, \odot, \mathbf{1} \rangle$ is a subsemiring of $\langle [0,1]^X, \wedge, \odot, \mathbf{1} \rangle$.

Continuous maps

Preimage of a function

Let X, Y be sets and $f : X \longrightarrow Y$ a map. If we identify the subsets of X and Y with their membership functions, the preimage of f is

$$f^{\leftarrow} : \chi \in \{0,1\}^Y \longmapsto \chi \circ f \in \{0,1\}^X.$$

Analogously, the fuzzy preimage of f is defined by

$$f^{\leftarrow\!\!\sim} : \chi \in [0,1]^Y \longmapsto \chi \circ f \in [0,1]^X.$$

MV-continuity

So, if $\langle X, \Omega_X\rangle$ and $\langle Y, \Omega_Y\rangle$ are MV-spaces, $f : X \longrightarrow Y$ is said to be MV-continuous if $f^{\leftarrow\!\!\sim}[\Omega_Y] \subseteq \Omega_X$.

Examples and bases

- $\langle X, \{\mathbf{0}, \mathbf{1}\}\rangle$ and $\langle X, [0,1]^X\rangle$ are MV-topological spaces.
- Any topology is an MV-topology.
- Let $d : X \longrightarrow [0, +\infty[$ be a metric on X and α a fuzzy point of X with support x. For any $r \in \mathbb{R}^+$, the open ball $B_r(\alpha)$ is
  $$B_r(\alpha)(y) := \begin{cases} \alpha(x) & \text{if } d(x,y) < r \\ 0 & \text{if } d(x,y) \geq r \end{cases}.$$
  The family of fuzzy subsets of X that are joins of open balls is an MV-topology on X that is said to be induced by d.

Definition

$\mathbf{T} = \langle X, \Omega\rangle \in {}^{\mathcal{MV}}\mathcal{T}\text{op}$. $B \subseteq \Omega$ is called a base for $\mathbf{T}$ if, for all $o \in \Omega$, $o = \bigvee_{i \in I} b_i$, with $\{b_i\}_{i \in I} \subseteq B$.

The shadow topology

Definition

For any MV-space $\mathbf{T} = \langle X, \Omega \rangle$, let $\mathrm{B}(\Omega) := \Omega \cap \{0,1\}^X$.
$\mathrm{Sh}\,\mathbf{T} = \langle X, \mathrm{B}(\Omega) \rangle$ is a topology in the classical sense, called the shadow of $\mathbf{T}$.

Sh is a functor

$\mathcal{T}$op is a full subcategory of ${}^{\mathcal{MV}}\mathcal{T}$op. The mapping $\mathrm{Sh} : {}^{\mathcal{MV}}\mathcal{T}\mathrm{op} \longrightarrow \mathcal{T}\mathrm{op}$ is a functor. It is, in fact, the left-inverse of the inclusion $\mathcal{T}\mathrm{op} \subseteq {}^{\mathcal{MV}}\mathcal{T}\mathrm{op}$.

The shadow of the MV-topology induced by a metric d is the topology induced by d.

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Compactness

A more complex situation

Due to the presence of two intersection and two union operations, compactness and each separation axiom can have at least two different MV-versions.

Compact spaces

An MV-space $\langle X, \Omega \rangle$ is said to be

- compact if any open covering of X contains an additive covering, i.e., for any $\Omega' \subseteq \Omega$ such that $\bigvee \Omega' = \mathbf{1}$, there exists a finite subset $\{o_1, \ldots, o_n\}$ of Ω' such that $o_1 \oplus \cdots \oplus o_n = \mathbf{1}$;
- strongly compact if any open covering of X contains a finite covering.

Separation

T_2 axioms

An MV-space $\mathbf{T} = \langle X, \Omega \rangle$ is called an Hausdorff (or separated, or T_2) space if, for any $x \neq y \in X$, there exist $o_x, o_y \in \Omega$ such that:

(i) $o_x(x) = o_y(y) = 1$,

(ii) $o_x(y) = o_y(x) = 0$,

(iii) $o_x \odot o_y = \mathbf{0}$.

$\mathbf{T}$ is said to be strongly separated if, for any $x \neq y \in X$, there exist $o_x, o_y \in \Omega$ satisfying (i) and

(iv) $o_x \wedge o_y = \mathbf{0}$.

T_2 definition do not need fuzzy points.

Stone MV-spaces

Remark

Strong separation implies separation and they both collapse to classical T_2 in the case of crisp topologies. The same holds for compactness.

Clopens and zero-dimensionality

Let $\mathbf{T} = \langle X, \Omega \rangle$ be an MV-space and $\Xi = \Omega^*$ be the family of closed fuzzy subsets. We denote by $\operatorname{Clop} \mathbf{T}$ the family $\Omega \cap \Xi$ of clopen fuzzy subsets of X. $\operatorname{Clop} \mathbf{T} \in \mathcal{MV}^{\mathrm{ss}}$, for any MV-space $\mathbf{T}$.

$\mathbf{T}$ is called zero-dimensional if $\operatorname{Clop} \mathbf{T}$ is a base for it.

Definition

A Stone MV-space is an MV-space which is compact, separated and zero-dimensional.

The MV-space $\langle \operatorname{Max} A, \Omega_A \rangle$

Remark

The category ${}^{\mathcal{MV}}\mathcal{S}$tone of Stone MV-spaces, with MV-continuous maps as morphisms, is a full subcategory of ${}^{\mathcal{MV}}\mathcal{T}$op.

The Maximal MV-spectrum

Let A be a semisimple MV-algebra. By Belluce representation theorem, there exists a canonical embedding $\iota : A \longrightarrow [0,1]^{\operatorname{Max} A}$. Then $\iota[A]$ generates, as a base, an MV-topology on $\operatorname{Max} A$. The family of open sets of such a space is denoted by Ω_A.
So, for any semisimple MV-algebra A, $\langle \operatorname{Max} A, \Omega_A \rangle$ denotes the MV-topological space on $\operatorname{Max} A$ having (an isomorphic copy of) A as a base.

A (proper) extension of Stone duality

Theorem

1. *The mappings*

$$\begin{array}{llllll} \Phi: & \mathbf{T} & \in & {}^{\mathcal{MV}}\mathcal{T}\mathrm{op} & \longmapsto & \mathrm{Clop}\,\mathbf{T} & \in & \mathcal{MV}^{\mathrm{ss}} \\ \Psi: & A & \in & \mathcal{MV}^{\mathrm{ss}} & \longmapsto & \langle \mathrm{Max}\,A, \Omega_A \rangle & \in & {}^{\mathcal{MV}}\mathcal{T}\mathrm{op} \end{array}$$

   define two contravariant functors.

2. *They yield a duality between $\mathcal{MV}^{\mathrm{ss}}$ and ${}^{\mathcal{MV}}\mathcal{S}\mathrm{tone}$, that is*
   - *for every semisimple MV-algebra A, ΨA is a Stone MV-space and A is isomorphic to the clopen algebra of such a space;*
   - *conversely, every Stone MV-space $\mathbf{T} = \langle X, \Omega \rangle$ is homeomorphic to $\Psi\Phi\mathbf{T}$.*

3. *The restriction of such a duality to Boolean algebras and Stone spaces coincide with the classical Stone duality.*

4. $\Phi\,\mathrm{Sh} = \mathrm{B}\,\Phi$ *and* $\Psi\,\mathrm{B} = \mathrm{Sh}\,\Psi$.

Graphically

$$
\begin{array}{ccc}
\mathcal{MV}^{\mathsf{ss}} & \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} & {}^{\mathcal{MV}}\mathcal{S}\mathsf{tone}^{\mathsf{op}} \\
\mathsf{B}\downarrow\ \uparrow\cup\!\!| & & \cup\!\!|\uparrow\ \downarrow\mathsf{Sh} \\
\mathcal{B}\mathsf{oole} & \underset{\Psi_{\restriction}}{\overset{\Phi_{\restriction}}{\leftrightarrows}} & \mathcal{S}\mathsf{tone}^{\mathsf{op}}
\end{array}
$$

Horizontal arrows: equivalences
Vertical arrows: inclusions of full subcategories and their left-inverses

Corollary

Strongly separated Stone MV-spaces are dual to hyperarchimedean MV-algebras.

Mundici equivalence

Unital Abelian ℓ-groups

Let $u\ell\mathcal{G}^{\mathrm{Ab}}$ be the category whose objects are Abelian lattice-ordered groups with a distinguished strong order unit and whose morphisms are unit-preserving ℓ-group homomorphisms.

Theorem [Mundici, 1986]

The categories $u\ell\mathcal{G}^{Ab}$ and $\mathcal{MV}$ are equivalent.

$\Gamma : u\ell\mathcal{G}^{\mathrm{Ab}} \longrightarrow \mathcal{MV}$ is defined as follows: for any Abelian $u\ell$-group $\langle G, +, -, \vee, \wedge, 0, u\rangle$, $\Gamma(G) = \langle [0, u], \oplus, ^*, 0\rangle$, where

- $x \oplus y := (x + y) \wedge u$,
- $x^* = u - x$.

We shall denote by Γ^{-1} the inverse of Γ.

$\Phi\Gamma$ and $\Gamma^{-1}\Psi$

Restrictions of Γ and Γ^{-1}

- $\mathcal{MV}^{\mathsf{ss}}$ is equivalent to the subcategory $u\ell\mathcal{G}_{\mathbb{R}}^{\mathsf{Ab}}$ of $u\ell\mathcal{G}^{\mathsf{Ab}}$ whose objects are, up to isomorphisms, $u\ell$-subgroups of a group of bounded real-valued functions from a set X, with $\mathbf{1}$ as unit.
- In such a restriction Boolean algebras correspond to subgroups of bounded integer-valued functions ($u\ell\mathcal{G}_{\mathbb{Z}}^{\mathsf{Ab}}$).

Corollary

Then $\Phi\Gamma$ and $\Gamma^{-1}\Psi$ form a duality between $u\ell\mathcal{G}_{\mathbb{R}}^{Ab}$ and ${}^{\mathcal{MV}}\mathcal{S}$tone. Such a duality obviously restricts to $u\ell\mathcal{G}_{\mathbb{Z}}^{Ab}$ and $\mathcal{S}$tone.

n-valued MV-algebras

$\mathcal{MV}_n$ and $\mathcal{B}R_n$

Let $n > 1$ be a natural number. In [Di Nola–Lettieri, 2000], the authors defined the category $\mathcal{B}R_n$:

- objects are pairs (B, R), where B is a Boolean algebra and R is an n-ary relation on B satisfying certain conditions;
- a morphism from (B, R) to (B', R') is a Boolean algebra homomorphism $f : B \longrightarrow B'$ such that $(a_1, \ldots, a_n) \in R$ implies $(f(a_1), \ldots, f(a_n)) \in R'$.

Now, let $\mathcal{MV}_n$ denote the subvariety $\mathcal{V}(S_n)$ of $\mathcal{MV}$ generated by the $(n+1)$-element chain $S_n = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$.

Future work

Theorem [Di Nola–Lettieri, 2000]

The categories $\mathcal{MV}_n$ and $\mathcal{BR}_n$ are equivalent.

$\mathcal{MV}_n \subseteq \mathcal{MV}^{\mathrm{ss}}$, for all $n > 1$.

${}^{\mathcal{MV}}\mathcal{S}$tone and $\mathcal{S}$tone

From an MV-topological viewpoint, $\mathcal{MV}_n$ is dual to the category of Stone MV-spaces of fuzzy sets with S_n-valued membership functions.
Next step will be to characterize a suitable category of Stone spaces with additional conditions which is dual to $\mathcal{BR}_n$ and, therefore, to $\mathcal{MV}_n$.

Further possible developments

A point-free approach

- MV-frames (reducts of complete BL-algebras?).
- Spatial and sober MV-frames.
- MV-topoi.

Applications

- Mathematical Morphology in digital image analysis.
- Geosystems.

Other ideas or suggestions are welcome. . .

Thank you!

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