Stone type dualities in nominal sets

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Overview

1. Boolean algebras and distributive lattices internal in nominal sets with restriction
2. Nominal topological spaces
3. Stone type duality theorems

Motivation

- Why nominal sets/nominal restriction sets?
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- A2: Possible applications
  Colagebraic logic in nominal sets. Coalgebras over nominal sets have been previously considered, for example to model π-calculus.
  Nominal domain theory in logical form, a la Abramsky?

Nominal Sets

We fix an infinite countable set of names $\mathbb{A}$. Let $\mathfrak{S}(\mathbb{A})$ denote the group of permutations on $\mathbb{A}$ generated by transpositions.

Definition

Consider a $\mathfrak{S}(\mathbb{A})$-action $(\mathbb{X}, \cdot)$ and an element $x \in \mathbb{X}$. We say that a set of names $A \subseteq \mathbb{A}$ *supports* x if for all $a, b \in \mathbb{A} \setminus A$ we have $(a\ b) \cdot x = x$.

We say that x is *finitely-supported* if there exists finite $A \subseteq \mathbb{A}$ that supports x.

Definition

A *nominal set* is a $\mathfrak{S}(\mathbb{A})$-action $(\mathbb{X}, \cdot)$ such that each element of $\mathbb{X}$ is finitely-supported.

Examples

1. The set of names can be equipped with the 'evaluation' action defined by
$$\pi \cdot a = \pi(a).$$
2. The finite and cofinite sets of atoms with the pointwise action
$$\pi \cdot X = \{\pi(x) \mid x \in X\}$$
3. The set of equivalence classes of lambda terms

The category of nominal sets Nom

- Finite limits are preserved by the forgetful functor $U : \mathrm{Nom} \to \mathrm{Set}$.
- Nom is cartesian closed: $[X, Y]$ is the nominal set of finitely supported functions wrt to the action

$$(\pi \cdot f)(x) = \pi \cdot_Y f(\pi^{-1} \cdot_X x)$$

- 2 with the trivial action is a subobject classifier.
- Nom is equivalent to a category of sheaves, i.e. the pullback preserving functors in $[\mathbb{I}, \mathrm{Set}]$.
- We have the power object functor $\mathcal{P} : \mathrm{Nom} \to \mathrm{Nom}^{op}$
$\mathcal{P}(X) = [X, 2]$

Internalizing Stone representation in Nom

A nominal Boolean algebra internal in Nom is a tuple $(B, \wedge, \neg)$ of a nonempty nominal set B, and equivariant functions
conjunction $\wedge : B \times B \to B$
negation $\neg : B \to B$
satisfying the usual axioms of BA.

A filter F for a nominal Boolean algebra B is a finitely-supported subset of B satisfying the usual properties.

The problem with the Ultrafilter Theorem in Nom: An arbitrary union of filters may no longer be finitely-supported!

Freshness

Given a nominal set X and $x \in X$ we say $a \# x$ iff $a \in \mathbb{A} \setminus \mathrm{supp}(x)$.

Definition

If $\phi(a, x_1, \ldots, x_n)$ is a formula describing properties of elements of a nominal set X and names in $\mathbb{A}$ we say $\text{И}a.\phi(a, x_1, \ldots, x_n)$ if $\{a \in \mathbb{A} \mid \phi(a, x_1, \ldots, x_n)\}$ is cofinite, or $\phi(a, x_1, \ldots, x_n)$ holds for cofinitely many a.

Theorem

The following are equivalent:

$\exists a \in \mathbb{A}. a \# x_1, \ldots, x_n \ \wedge \phi(a, x_1, \ldots, x_n)$.

$\forall a \in \mathbb{A}. a \# x_1, \ldots, x_n \ \Rightarrow \phi(a, x_1, \ldots, x_n)$.

$\text{И}a.\phi(a, x_1, \ldots, x_n)$.

$$\text{И}a.(\phi \wedge \psi) \text{ iff } \text{И}a.\phi \wedge \text{И}a.\psi$$

$$\text{И}a.\neg\phi \text{ iff } \neg\text{И}a.\phi$$

Restriction on power objects

Definition
If $Y \in \mathcal{P}(X)$ then define

$$\mathsf{n}a.Y = \{x \mid \mathsf{И}b.(b\ a) \cdot x \in Y\}.$$

Connection with И

$$\mathsf{И}a.\phi(a) \text{ if and only if } a \in \mathsf{n}a.\{x \in \mathbb{A} \mid \phi(x)\}.$$

Axioms for restriction

$$\begin{array}{lrl}
(\textbf{Swap}) & & \text{И}a.\text{И}b.X = \text{И}b.\text{И}a.X \\
(\textbf{Garbage}) & a\#X \Rightarrow & \text{И}a.X = X \\
(\textbf{Alpha}) & b\#X \Rightarrow & \text{И}a.X = \text{И}b.(b\ a)\cdot X \\
\\
(\textbf{Distrib}) & & \text{И}a.(X \wedge Y) = (\text{И}a.X) \wedge (\text{И}a.Y) \\
(\textbf{SelfDual}) & & \neg\text{И}a.X = \text{И}a.\neg X
\end{array}$$

Nominal sets and nominal Boolean algebras with restriction

Definition

A nominal restriction set is a nominal set X equipped with a restriction operation $\text{и} : \mathbb{A} \times X \to X$ that satisfies:

$$
\begin{array}{lrl}
(\textbf{Swap}) & & \text{и}a.\text{и}b.x = \text{и}b.\text{и}a.x \\
(\textbf{Garbage}) & a\#x \Rightarrow & \text{и}a.x = x \\
(\textbf{Alpha}) & b\#x \Rightarrow & \text{и}a.x = \text{и}b.(b\ a) \cdot x
\end{array}
$$

Nominal sets and nominal Boolean algebras with restriction

Definition

A nominal Boolean algebra with restriction is a nominal Boolean algebra B equipped with a restriction operation $\text{и} : \mathbb{A} \times B \to B$ that satisfies:

$$
\begin{array}{llr@{\;}l}
(\textbf{Swap}) & & \text{и}a.\text{и}b.x = & \text{и}b.\text{и}a.x \\
(\textbf{Garbage}) & a\#x \Rightarrow & \text{и}a.x = & x \\
(\textbf{Alpha}) & b\#x \Rightarrow & \text{и}a.x = & \text{и}b.(b\ a) \cdot x \\
(\textbf{Distrib}) & & \text{и}a.(x \wedge y) = & (\text{и}a.x) \wedge (\text{и}a.y) \\
(\textbf{SelfDual}) & & \neg\text{и}a.x = & \text{и}a.\neg x
\end{array}
$$

n-filters

An n-filter of a nominal Boolean algebra with restriction is a finitely-supported subset $F \in \mathcal{P}(B)$ s.t.

1. $\bot \notin F$
2. $x \in F$ and $y \in F$ iff $x \wedge y \in F$
3. $\text{И}a.\forall x \in B. x \in F \Leftrightarrow \text{и}a.x \in F$.

Key lemma: An n-filter is maximal iff it is maximal amongst n-filters with smaller support.

Nominal topologies

Definition

A nominal topological space T is a pair $(T, \mathcal{O}_T)$ of a nominal set T and equivariant set of open sets $\mathcal{O}_T \subseteq \mathcal{P}(T)$ such that:

- $\emptyset \in \mathcal{O}_\mathsf{T}$ and $T \in \mathcal{O}_\mathsf{T}$
- $U \in \mathcal{O}_T \wedge V \in \mathcal{O}_T$ implies $U \cap V \in \mathcal{O}_T$.
- $\mathcal{U} \in \mathcal{P}(\mathcal{O}_T)$ implies $\bigcup \mathcal{U} \in \mathcal{O}_T$; we call this a finitely-supported union.

Morphisms of nominal topological spaces are continuous equivariant maps.

Nominal topology on Ultr(B)

The maximal n-filters of a nominal Boolean algebra with restriction forms a nominal subset Ultr(B) of $\mathcal{P}(B)$.
Consider the nominal topology on Ultr(B) generated by the finitely-supported sets

$$\{F \in \mathrm{Ultr}(B) \mid x \in F\}$$

for all $x \in B$.

From $\mathsf{nBA}_{\text{И}}$ to nominal topological spaces

We have a contravariant functor $S : \mathsf{nBA}_{\text{И}} \to \mathsf{nTop}$ defined by

$$B \mapsto (\mathsf{Ultr}(B), \mathcal{O}_{\mathsf{Ultr(B)}}).$$

and for $f : B \to B'$ we have $S(f)(F') = f^{-1}(F')$ for a maximal n-filter F' of B'

S is faithful.

Nominal Stone spaces with restriction

... is a nominal topological space $(T, \mathcal{O}_T)$ that is:

1. totally-separated
2. n-compact:
   A finitely-supported subset $\mathcal{U} \subseteq \mathcal{O}_T$ is called n-stable when
   $$\text{И}a.\forall U.(U \in \mathcal{U} \Rightarrow \mathsf{n}a.U \in \mathcal{U}).$$
   A nominal topological space is n-compact when every n-stable open cover has a finite subcover.
3. For all $a \in \mathbb{A}$, $U \in \mathcal{O}_T$ clopen implies $\mathsf{n}a.U \in \mathcal{O}_T$.

A duality theorem

Theorem

The categories of nominal Boolean algebras with restriction and nominal Stone spaces are dually equivalent.

Nominal distributive lattices with restriction

A nominal distribuitive lattice with restriction is a nominal distributive lattice equipped with a restriction operation $\text{И} : \mathbb{A} \times B \to B$ that satisfies:

$$
\begin{array}{llr}
(\textbf{Swap}) & & \text{И}a.\text{И}b.x = \text{И}b.\text{И}a.x \\
(\textbf{Garbage}) & a\#x \Rightarrow & \text{И}a.x = x \\
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(\textbf{Distrib}) & & \text{И}a.(x \wedge y) = (\text{И}a.x) \wedge (\text{И}a.y) \\
(\textbf{SelfDual}) & & \text{И}a.(x \vee y) = (\text{И}a.x) \vee (\text{И}a.y)
\end{array}
$$

Dualities for distributive lattices

The following three categories are isomorphic and dually equivalent to DL:

1. spectral spaces
2. Priestly spaces
3. pairwise Stone spaces

Duality for nominal distributive lattices with restriction

A nominal bitopological space, that is a is a nominal set X equipped with two nominal topologies τ_1 and τ_2, is called nominal pairwise Stone space when it is

1. pairwise Haussdorff: for all distinct $x, y \in X$ there exists $U \in \tau_1$ and $V \in \tau_2$, disjoint such that $x \in U$ and $y \in V$, or there exists $U \in \tau_2$ and $V \in \tau_1$, disjoint such that $x \in U$ and $y \in V$.
2. pairwise zero-dimensional: the sets that are open in τ_1 and closed in τ_2 form a basis for τ_1 and the sets that are open in τ_2 and closed in τ_1 form a basis for τ_2.
3. pairwise n-compact: for all n-stable $\mathcal{U}_1 \in \mathcal{P}(\tau_1)$ and n-stable $\mathcal{U}_2 \in \mathcal{P}(\tau_2)$ such that $\bigcup \mathcal{U}_1 \cup \bigcup \mathcal{U}_2$ covers X there exists a finite subset of $\mathcal{U}_1 \cup \mathcal{U}_2$ that covers X.
4. pairwise n-closed: if for all $a \in \mathbf{A}$ and $U \in \tau_1$ closed in τ_2 we have $\mathsf{n}a.U \in \tau_1$ and for all $a \in \mathbf{A}$ and $U \in \tau_2$ closed in τ_1 we have $\mathsf{n}a.U \in \tau_2$

Duality for nominal distributive lattices with restriction

Theorem

The categories of nominal distributive lattices with restriction and nominal pairwise Stone spaces are dually equivalent.

Conclusions and future work

We have seen Stone type dualities for nominal Boolean algebras/distributive lattices with restriction.

Is restriction really necessary?

Applications ...