Craig interpolation in displayable logics

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Craig interpolation

Definition

A (propositional) logic satisfies Craig interpolation iff for any provable entailment $F \vdash G$ there exists an interpolant I s.t.:

$$F \vdash I \text{ provable and } I \vdash G \text{ provable and } \mathcal{V}(I) \subseteq \mathcal{V}(F) \cap \mathcal{V}(G)$$

($\mathcal{V}(X)$ is the set of propositional variables occurring in X)

Craig interpolation has applications in:

- logic: consistency; compactness; definability
- computer science: invariant generation; type inference; model checking; decomposition of complex ontologies

Display calculi

- Consecution calculi à la Gentzen with left- and right-introduction rules for logical connectives;
- Characterised by: any part of a consecution can be "displayed" alone on one side of the $\vdash$;
- Uses a richer consecution structure than simple sequents;
- Main advantage: cut-elimination is guaranteed when the proof rules satisfy some simple well-formedness conditions;
- But decidability, interpolation etc. don't follow from cut-elimination as they often do in sequent calculi.
- We show Craig interpolation for a large class of display calculi.

Display calculus syntax

- Formulas given by:

$$F ::= P \mid \top \mid \bot \mid \neg F \mid F \& F \mid F \vee F \mid F \rightarrow F \mid \ldots$$

- Structures given by:

$$X ::= F \mid \emptyset \mid \sharp X \mid X \,;\, X$$

- Consecutions are given by $X \vdash Y$ for X, Y structures.
- We classify substructures of $X \vdash Y$ as antecedent or consequent parts (similar to positive / negative occurrences in formulas).

Display-equivalence

We have the following display postulates:

$$\begin{array}{ccccc} X ; Y \vdash Z & <>_D & X \vdash \sharp Y ; Z & <>_D & Y ; X \vdash Z \\ X \vdash Y ; Z & <>_D & X ; \sharp Y \vdash Z & <>_D & X \vdash Z ; Y \\ X \vdash Y & <>_D & \sharp Y \vdash \sharp X & <>_D & \sharp\sharp X \vdash Y \end{array}$$

Display-equivalence $\equiv_D$ given by transitive closure of $<>_D$.

Proposition (Display property)

For any antecedent / consequent part Z of $X \vdash Y$ there is a W s.t. $X \vdash Y \equiv_D Z \vdash W$ / $X \vdash Y \equiv_D W \vdash Z$.

Some proof rules

Identity rules:

$$\frac{}{F \vdash F}\,(\mathrm{Id}) \qquad \frac{X' \vdash Y'}{X \vdash Y}\ (X \vdash Y \equiv_D X' \vdash Y')\ \ (\equiv_D)$$

Logical rules:

$$\frac{F\,;G \vdash X}{F\&G \vdash X}\,(\&\mathrm{L}) \qquad \frac{X \vdash F \quad Y \vdash G}{X\,;Y \vdash F\&G}\,(\&\mathrm{R}) \qquad \ldots$$

Structural rules:

$$\frac{W\,;(X\,;Y) \vdash Z}{(W\,;X)\,;Y \vdash Z}\,(\alpha) \qquad \frac{\emptyset\,;X \vdash Y}{X \vdash Y}\,(\emptyset \mathrm{C_L})$$

$$\frac{X \vdash Z}{X\,;Y \vdash Z}\,(\mathrm{W}) \qquad \frac{X\,;X \vdash Y}{X \vdash Y}\,(\mathrm{C}) \qquad \ldots$$

Interpolation: our approach

- Proof-theoretic strategy: given a cut-free proof of $X \vdash Y$, we construct its interpolant I.
- Induction on the proof of $X \vdash Y$: given interpolants for the premises of a rule we need to construct an interpolant for the conclusion.
- Problem: not enough information to do this for display steps, e.g.:

$$\frac{X \; ; Y \vdash Z}{X \vdash \sharp Y \; ; Z} \, (\equiv_D)$$

Local AD-interpolation (LADI) property

Let $\equiv_{AD}$ be the least equivalence closed under $\equiv_D$ and applications of associativity (α) (if present).

Definition

A proof rule with conclusion $\mathcal{C}$ has the LADI property if, given that for each premise of the rule $\mathcal{C}_i$ we have interpolants for all $\mathcal{C}'_i \equiv_{AD} \mathcal{C}_i$, we can construct interpolants for all $\mathcal{C}' \equiv_{AD} \mathcal{C}$.

Proposition

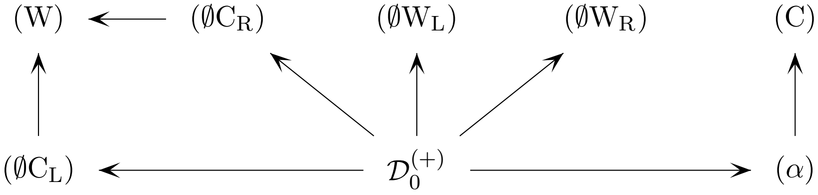
If the proof rules of a (cut-free) display calculus $\mathcal{D}$ all have the LADI property then $\mathcal{D}$ has the Craig interpolation property.

LADI: (&R)

$$\frac{X \vdash F \quad Y \vdash G}{X \,;\, Y \vdash F\&G}\ (\&\mathrm{R})$$

Need interpolant for arbitrary $W \vdash Z \equiv_{AD} X;Y \vdash F\&G$.
Case: $F\&G$ occurs in Z.
Subcase: W built entirely from parts of X ($W \lhd X$).
By a LEMMA $\exists U.\ X \vdash F \equiv_{AD} W \vdash U$.
Claim: interpolant I for $W \vdash U$ is an interpolant for $W \vdash Z$.
Main issue: show $I \vdash Z$ provable given $I \vdash U$ provable.

Summary of results



LADI of the proof rule(s) at a node holds in a calculus with all of the proof rules at its ancestor nodes. Thus:

Theorem

Any display calculus satisfying the constraints in the above diagram has Craig interpolation.

Future work

1. Machine formalisation of results; an Isabelle mechanisation, led by Jeremy Dawson (ANU), is currently under way.
2. More logics:
   - non-commutative logics;
   - multiple-family display calculi (bunched & relevant logics);
   - modalities, quantifiers, linear exponentials . . .

Further reading

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