Relation lifting on preorders, metric spaces, etc.

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joint work with
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The characterisation theorem (M. Barr, 1970)

For a functor $T : \mathsf{Set} \longrightarrow \mathsf{Set}$, the following are equivalent:

1. There is a 2-functor $\overline{T} : \mathsf{Rel}(\mathsf{Set}) \longrightarrow \mathsf{Rel}(\mathsf{Set})$ such that the square

$$\begin{array}{ccc} \mathsf{Rel}(\mathsf{Set}) & \overset{\overline{T}}{\dashrightarrow} & \mathsf{Rel}(\mathsf{Set}) \\ {\scriptstyle (-)_\diamond}\uparrow & & \uparrow{\scriptstyle (-)_\diamond} \\ \mathsf{Set} & \underset{T}{\longrightarrow} & \mathsf{Set} \end{array}$$

   commutes.
2. T preserves weak pullbacks.

Here, for $f : A \longrightarrow B$, $f_\diamond(b, a) = 1$ iff $b = fa$.

Why is this interesting?

The semantics of coalgebraic cover modality ∇, for $T : \mathsf{Set} \longrightarrow \mathsf{Set}$

1. The modal language $\mathcal{L}$

$$\varphi ::= p \mid \top \mid (\varphi \wedge \varphi) \mid (\neg\varphi) \mid \nabla\alpha$$

   for $p \in \mathsf{At}$, $\alpha \in T\mathcal{L}$.

2. Semantics in a coalgebra $c : X \longrightarrow TX$. Define

$$x \Vdash \nabla\alpha \quad \text{iff} \quad c(x)\ \overline{T}(\Vdash)\ \alpha$$

   for every $x \in X$, $\alpha \in T\mathcal{L}$.

Moss, Kurz, Kupke, Venema, Bílková, Palmigiano,...

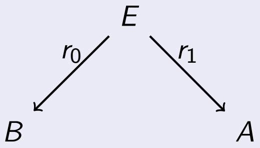
All Kripke polynomial set functors T admit such a 2-functorial extension $\overline{T}$. Hence there are "well-behaved" proof systems for ∇.

Definition

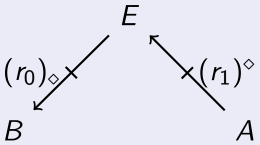
A relation from A to B is a map $R : B \times A \longrightarrow 2$, denoted by

$$R : A \longrightarrow\!\!\!\!\!\!\!+\;\;\; B$$

Relation R is tabulated by the span



if $R =$



where $(r_0)_\diamond(b, e) = 1$ iff $b = r_0(e)$, $(r_1)^\diamond(e, a) = 1$ iff $r_1(e) = a$.

Weak pullbacks

A square $\begin{array}{ccc} \mathcal{P} & \xrightarrow{p_1} & \mathcal{B} \\ {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle g} \\ \mathcal{A} & \xrightarrow[f]{} & \mathcal{C} \end{array}$ in Set is a weak pullback

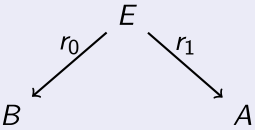
iff the square $\begin{array}{ccc} \mathcal{P} & \xleftarrow{(p_1)^\diamond} & \mathcal{B} \\ {\scriptstyle (p_0)_\diamond}\downarrow & & \downarrow{\scriptstyle (g)_\diamond} \\ \mathcal{A} & \xleftarrow[(f)^\diamond]{} & \mathcal{C} \end{array}$ commutes in Rel(Set)

or, equivalently, iff for every a and b

$fa = gb$ iff there exists w s.t. $a = p_0(w)$ and $p_1(w) = b$.

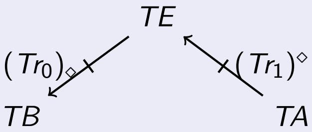
Definition of $\overline{T}$

Suppose $R : A \longrightarrow\!\!\!\!\!\!\!+\;\; B$ is tabulated by



Define $\overline{T}(R) : TA \longrightarrow\!\!\!\!\!\!\!+\;\; TB$

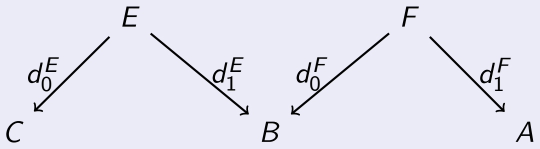
by putting



$$\overline{T}(R)(\beta,\alpha) = \bigvee_{w} (\beta = Tr_0(w)) \wedge (Tr_1(w) = \alpha)$$

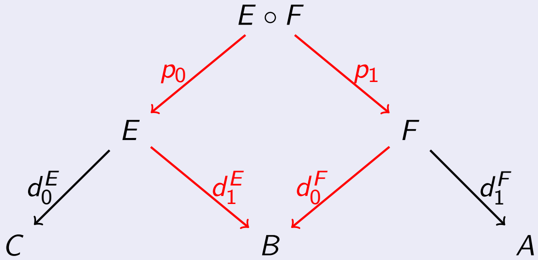
How to compose two relations:

tabulate the relations...



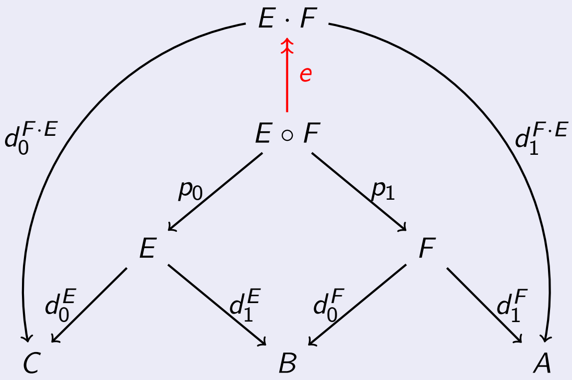
How to compose two relations:

…form the pullback…

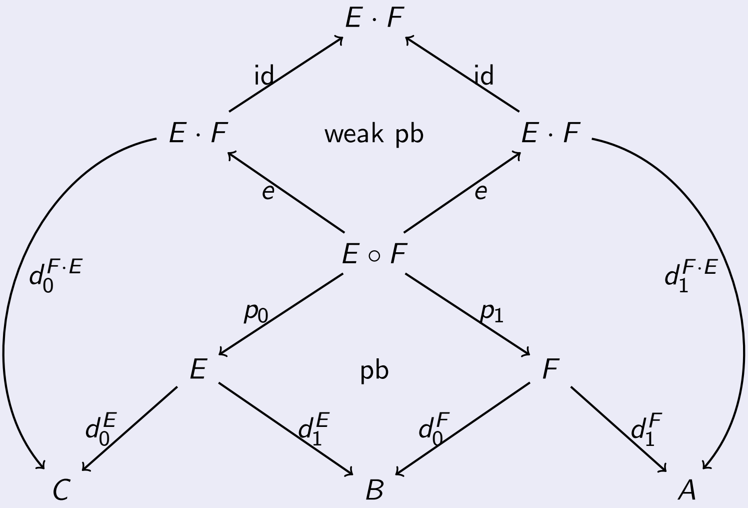


How to compose two relations:

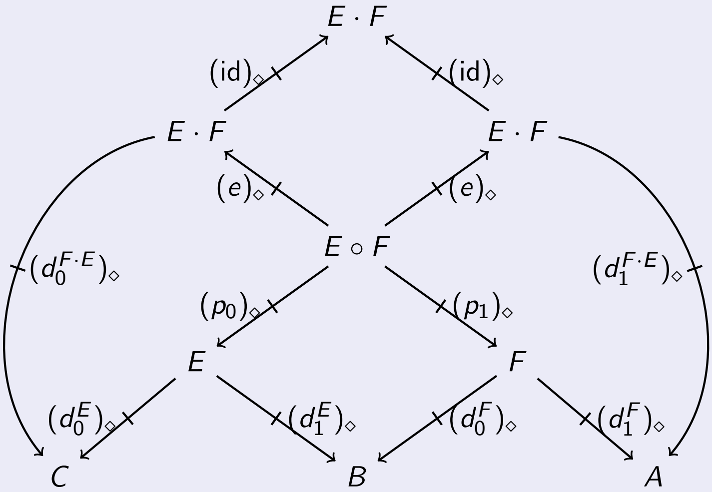
...form the quotient...



The composition diagram written more carefully



The presence of (weak) pullbacks in Set makes the following commutative in Rel(Set)



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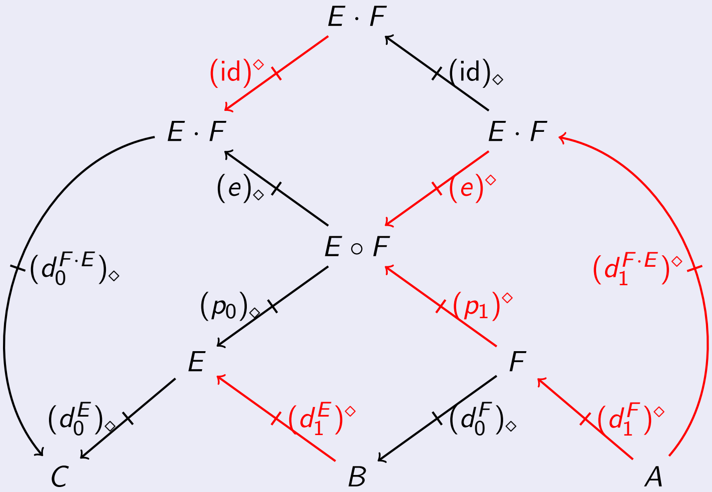

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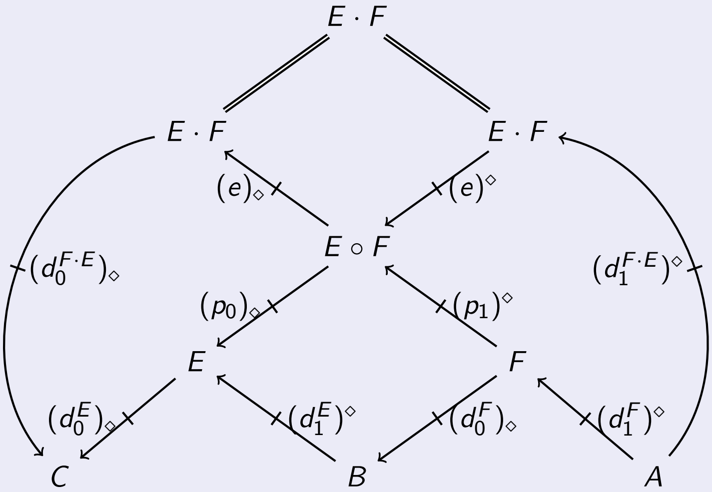

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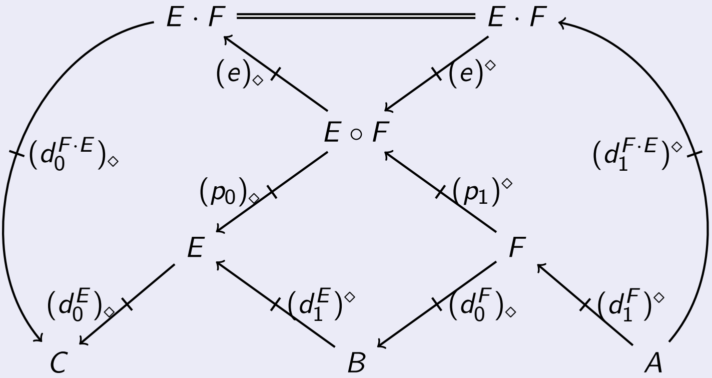
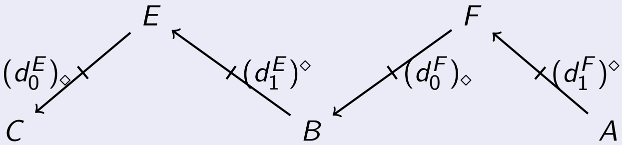


The previous makes composition work smoothly



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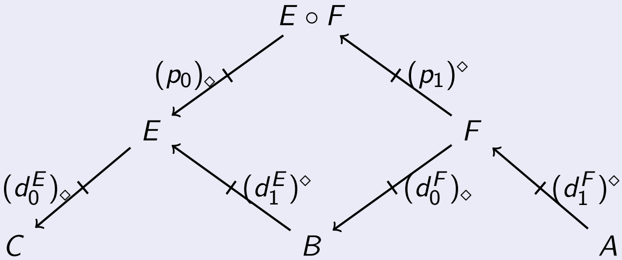

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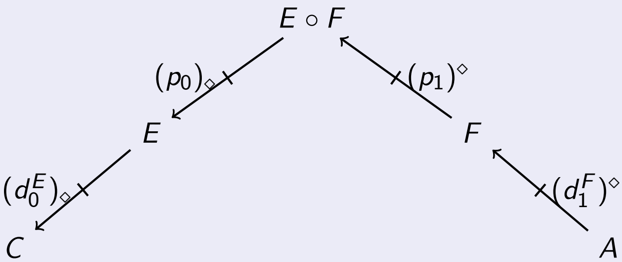

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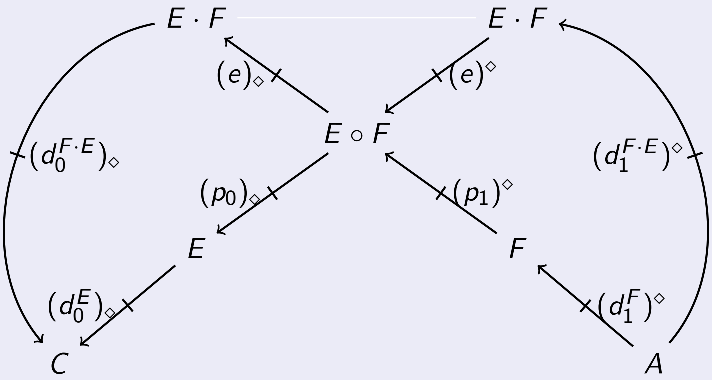

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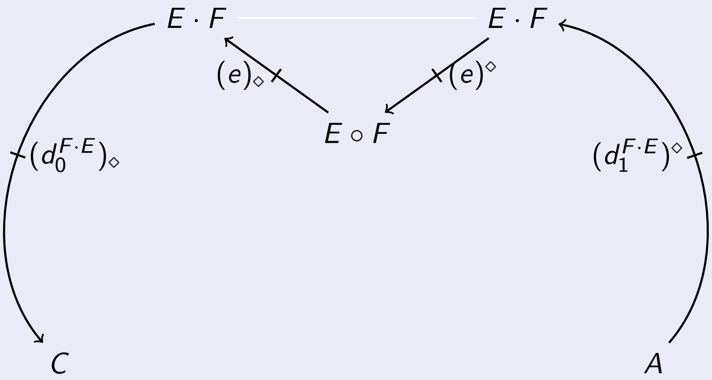

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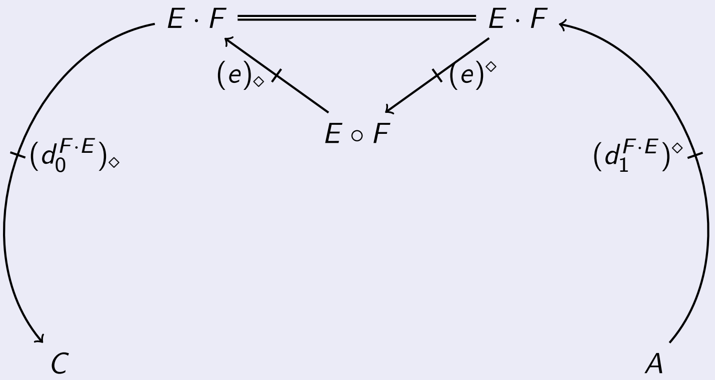

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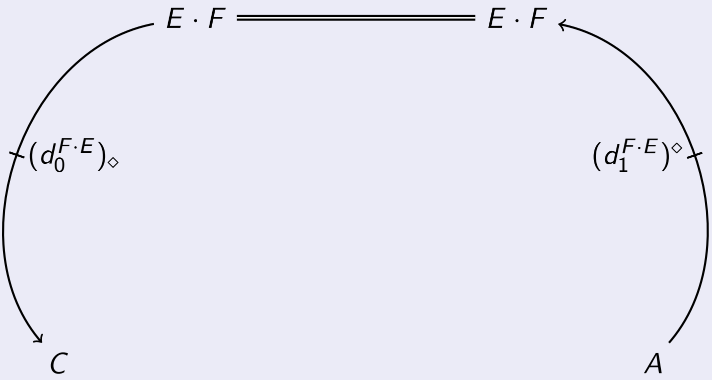
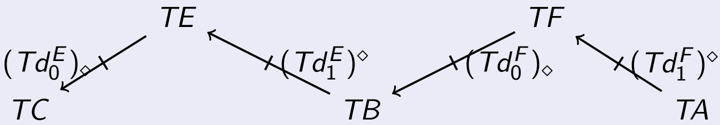


And if T preserves weak pullbacks, $\overline{T}$ preserves composition



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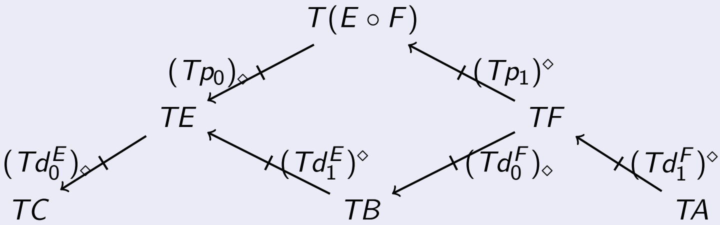

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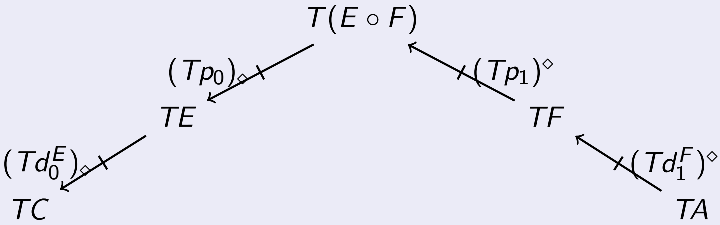

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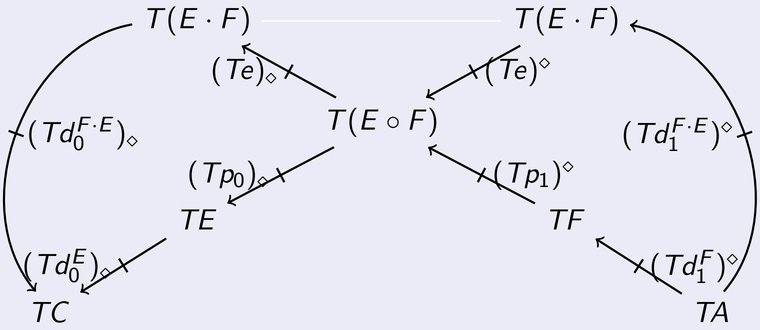

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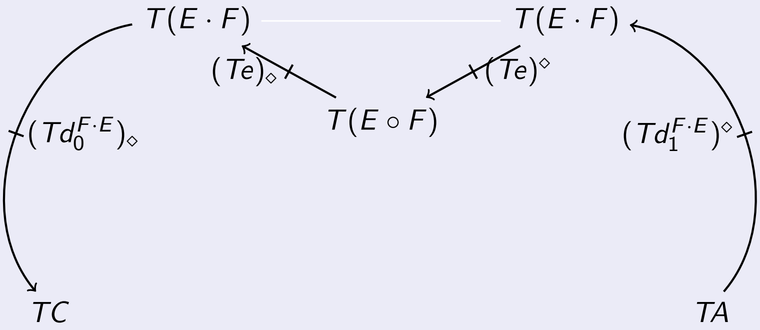

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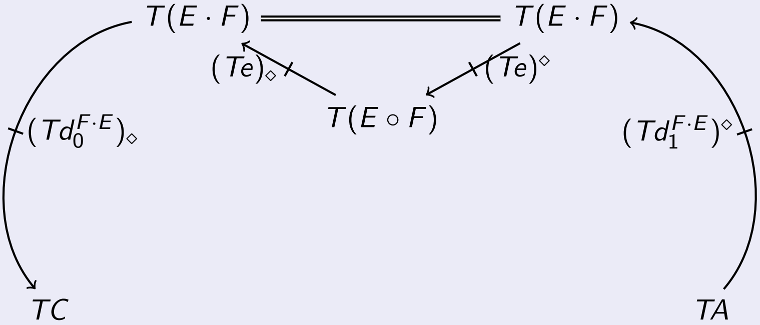

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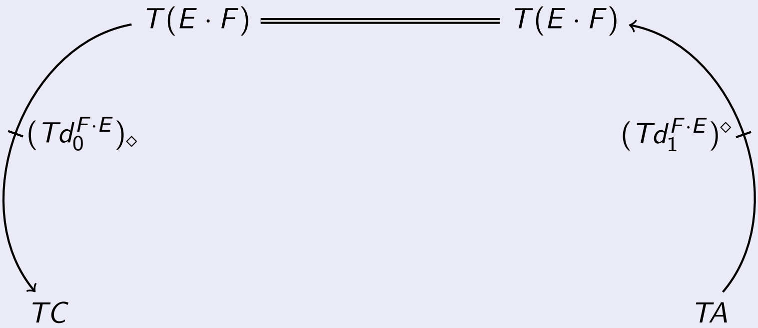


We want to pass from Set to more general categories to obtain more general applications.

The level of generality:

Set is replaced by $\mathcal{V}$-cat, $\mathcal{V}$ being rather simple.

Problem:

"Relations" can no longer be tabulated by spans, we need to cotabulate them by cospans.

Advantages:

1. Hermida's idea goes through with only small modifications.
2. All "Kripke-polynomial" functors on $\mathcal{V}$-cat admit a functorial relation lifting.

Definition

A commutative quantale[a] $\mathcal{V}$ is a tuple $(\mathcal{V}_o, \otimes, I, [-,-])$ where

1. $\mathcal{V}_o$ is a complete lattice.
2. The tensor $\otimes$ is commutative, associative, has I as a unit.
3. There is an adjunction $- \otimes a \dashv [a,-] : \mathcal{V}_o \longrightarrow \mathcal{V}_o$, i.e., $x \otimes a \leq y$ iff $x \leq [a,y]$ holds, for every a, x and y.

[a] Or, a commutative complete residuated lattice.

Examples

1. $\mathcal{V}_o$ = two-element chain, $\otimes$ = meet, I = top.
2. $\mathcal{V}_o$ = unit interval with reversed order, $\otimes$ = max, I = zero.
3. …many others.

Definition

A small $\mathcal{V}$-category $\mathcal{A}$ consists of a small set of objects, a, b, ..., and $\mathcal{A}(a, b)$ in $\mathcal{V}_o$, for every pair a, b of objects, such that

1. $I \leq \mathcal{A}(a, a)$, for every a.
2. $\mathcal{A}(a, b) \otimes \mathcal{A}(b, c) \leq \mathcal{A}(a, c)$, for every a, b, c.

A $\mathcal{V}$-functor $f : \mathcal{A} \longrightarrow \mathcal{B}$ consists of an object-assignment $a \mapsto fa$ such that $\mathcal{A}(a, b) \leq \mathcal{B}(fa, fb)$ holds, for every a, b.

Small $\mathcal{V}$-categories and $\mathcal{V}$-functors form a 2-category

$$\mathcal{V}\text{-cat}$$

The 2-cell $f \rightarrow g$ witnesses the inequality $I \leq \bigwedge_x \mathcal{B}(fx, gx)$.

Examples

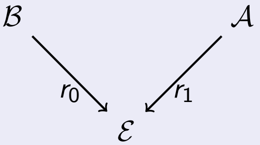
1. $\mathcal{V}_o$ = two-element chain, $\otimes$ = meet, I = top. Then $\mathcal{V}$-cat = preorders and monotone maps.
2. $\mathcal{V}_o$ = unit interval with reversed order, $\otimes$ = max, I = zero. Then $\mathcal{V}$-cat = ultrametric spaces and nonexpanding maps.
3. ... many others.

Definition

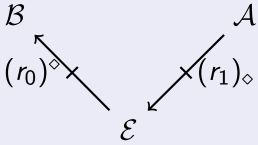
A relation[a] from $\mathcal{A}$ to $\mathcal{B}$ is a $\mathcal{V}$-functor $R : \mathcal{B}^{op} \otimes \mathcal{A} \longrightarrow \mathcal{V}$,

denoted by $R : \mathcal{A} \longrightarrow\!\!\!\!\!\!\!\!\mapsto \mathcal{B}$

Relation R is cotabulated by the cospan



if $R =$



where $(r_1)_\diamond(e, a) = \mathcal{E}(e, r_1(a))$, $(r_0)^\diamond(b, e) = \mathcal{E}(r_0(b), e)$.

[a] Or, module, or, profunctor, or, distributor.

Street's characterisation of relations in $\mathcal{V}$-cat (1980)

Relations in $\mathcal{V}$-cat correspond to cospans that are codiscrete cofibrations in $\mathcal{V}$-cat.
Composition of these cospans involves pushouts in $\mathcal{V}$-cat and fully-faithful $\mathcal{V}$-functors.
$\mathcal{V}$-functor $f : \mathcal{A} \longrightarrow \mathcal{B}$:

$$\mathcal{A}(a,b) \leq \mathcal{B}(fa,fb)$$

Fully-faithful $\mathcal{V}$-functor $f : \mathcal{A} \longrightarrow \mathcal{B}$:

$$\mathcal{A}(a,b) = \mathcal{B}(fa,fb)$$

(Weak) pullbacks are replaced by exact squares

A lax square $\begin{array}{ccc} \mathcal{P} & \xrightarrow{p_1} & \mathcal{B} \\ {\scriptstyle p_0}\downarrow & \nearrow & \downarrow{\scriptstyle g} \\ \mathcal{A} & \xrightarrow[f]{} & \mathcal{C} \end{array}$ in $\mathcal{V}$-cat is exact

iff the square $\begin{array}{ccc} \mathcal{P} & \xleftarrow{(p_1)^\diamond} & \mathcal{B} \\ {\scriptstyle (p_0)_\diamond}\downarrow & & \downarrow{\scriptstyle (g)_\diamond} \\ \mathcal{A} & \xleftarrow[(f)^\diamond]{} & \mathcal{C} \end{array}$ commutes in $\mathsf{Rel}(\mathcal{V}\text{-cat})$

iff, for all a and b

$$\mathcal{C}(fa, gb) = \bigvee_w \mathcal{A}(a, p_0(w)) \otimes \mathcal{B}(p_1(w), b).$$

The characterisation theorem

For a 2-functor $T : \mathcal{V}\text{-cat} \longrightarrow \mathcal{V}\text{-cat}$, the following are equivalent:

1. There is a 2-functor $\overline{T} : \mathsf{Rel}(\mathcal{V}\text{-cat}) \longrightarrow \mathsf{Rel}(\mathcal{V}\text{-cat})$ such that the square

$$\begin{array}{ccc} \mathsf{Rel}(\mathcal{V}\text{-cat}) & \overset{\overline{T}}{\dashrightarrow} & \mathsf{Rel}(\mathcal{V}\text{-cat}) \\ {\scriptstyle (-)_\diamond}\uparrow & & \uparrow{\scriptstyle (-)_\diamond} \\ \mathcal{V}\text{-cat} & \underset{T}{\longrightarrow} & \mathcal{V}\text{-cat} \end{array}$$

   commutes.
2. T preserves exact squares.

Here, for $f : \mathcal{A} \longrightarrow \mathcal{B}$, $f_\diamond(b, a) = \mathcal{B}(b, fa)$.

Definition of $\overline{T}$

Suppose $R : \mathcal{A} \longrightarrow\!\!\!\!\!\!\!\!+\;\;\;\; \mathcal{B}$ is cotabulated by

$$\begin{array}{ccc} \mathcal{B} & & \mathcal{A} \\ & \searrow^{r_0} \quad {}^{r_1}\swarrow & \\ & \mathcal{E} & \end{array}$$

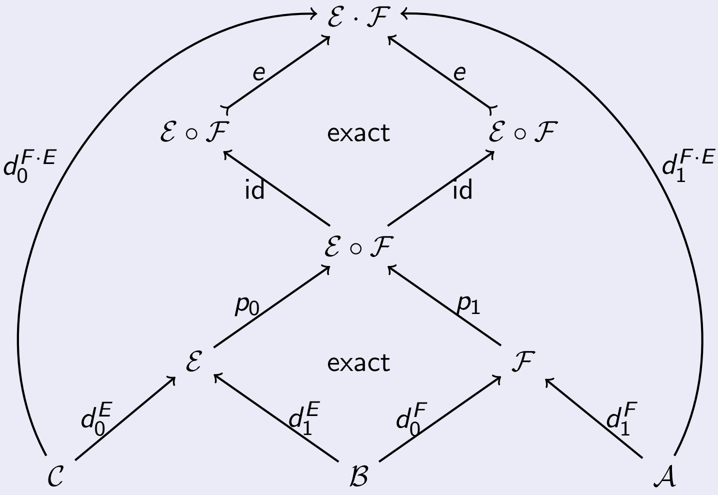
Define $\overline{T}(R) : T\mathcal{A} \longrightarrow\!\!\!\!\!\!\!\!+\;\;\;\; T\mathcal{B}$

as the composite

$$\begin{array}{ccc} T\mathcal{B} & & T\mathcal{A} \\ & {}^{(Tr_0)^\diamond}\nwarrow \quad \swarrow^{(Tr_1)_\diamond} & \\ & T\mathcal{E} & \end{array}$$

$$\begin{aligned} \overline{T}(R)(\beta, \alpha) &= \bigvee_w T\mathcal{E}(w, Tr_1(\alpha)) \otimes T\mathcal{E}(Tr_0(\beta), w) \\ &= T\mathcal{E}(Tr_0(\beta), Tr_1(\alpha)) \end{aligned}$$

The composition diagram



And the rest of the reasoning is analogous to sets.

Kripke-polynomial functors

All 2-functors $T : \mathcal{V}\text{-cat} \longrightarrow \mathcal{V}\text{-cat}$, given by

$$T ::= \mathrm{Id} \mid \mathrm{const}_{\mathcal{X}} \mid T + T \mid T \times T \mid T \otimes T \mid T^{\partial} \mid \mathcal{X} \mapsto [\mathcal{X}^{op}, \mathcal{V}]$$

where $T^{\partial}\mathcal{X} = (T(\mathcal{X}^{op}))^{op}$, preserve exact squares. Hence they give rise to a "well-behaved" coalgebraic cover modality.

The case of preorders

The explicit description of relation liftings is in

> M. Bílková, A. Kurz, D. Petrișan and J. V., Relation liftings on preorders and posets, to appear in CALCO 2011

Quoted references

1. M. Barr, Relational algebras, Reports of the Midwest Category Seminar IV, Lecture Notes in Mathematics, 1970, Volume 137 (1970), 39–55.
2. C. Hermida, A categorical outlook on relational modalities and simulations, *preprint*, http://maggie.cs.queensu.ca/chermida/
3. R. Street, Fibrations in bicategories, *Cahiers de Top. et Géom. Diff.* XXI.2 (1980), 111–159