Some Algebra for (Flat) Modal Fixpoint Logics

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Outline

Context

Flat Fixpoint Logics

Flat fixpoint logics – as extensions of $\mathbb{K}$

... – as fragments of $\mathcal{L}_\mu$

Algebraic logic perspective

Propositional modal logic as an algebraic system

The nonsense path to completeness

Three key properties

Residuation and constructiveness

Regularity

Conclusions

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Igor's completeness proof

Original goal : *explain* Walukiewicz's Theorem :

Theorem (Walukiewicz 2000)

Kozen's axiomatization of the modal μ-calculus is complete
(w.r.t. the semantics of Kripke frames)

to some non-standard (for the time) audience :

- Yde, myself,
- proof theorists,
- modal logicians,
- algebraists,
- poset and category theorists ...

Some previous work

D. Kozen.
Results on the propositional μ-calculus.
Theoret. Comput. Sci., 27(3):333–354, 1983.

I. Walukiewicz.
Completeness of Kozen's axiomatisation of the propositional μ-calculus.
Inform. and Comput., 157(1-2):142–182, 2000.
LICS 1995 (San Diego, CA).

Our contributions

Our (partial) explanation and digestion of the theorem :

- Luigi Santocanale.
  Completions of μ-algebras.
  Annals of Pure and Applied Logic, 154(1):27–50, May 2008.
- Luigi Santocanale and Yde Venema.
  Completeness for flat modal fixpoint logics.
  Annals of Pure and Applied Logic, 162(1):55–82, 2010.
- Yde Venema and Luigi Santocanale.
  Uniform interpolation for monotone modal logic.
  In Lev Beklemishev, Valentin Goranko, and Valentin Shehtman, editors, *Advances in Modal Logic, Volume 8*, pages 350–370. College Publications, 2010.

Outline

A slogan on logics of programs

Def. A *logic of programs* is :

[multi]modal propositional logic $\mathbb{K}$

$+$ some extremal fixpoints μ, ν .

An example, $\mathbb{CTL}$:

$$\mathcal{L}(\mathbb{CTL}) = \mathcal{L}(\mathbb{K}) + \{ \, \mathcal{U} \, \}$$

Meaning of $\mathcal{U}$:

$$\begin{aligned} p\mathcal{U}q &\equiv \text{“}p \text{ holds until } q\text{”} \\ &\equiv LFP \text{ of } \gamma(x, p, q) = q \vee (p \wedge \Box x) \,. \end{aligned}$$

Flat fixpoint logics

Let

$$\Gamma = \{ \gamma_i(x) \mid i = 1, \ldots, n \},$$

where $\gamma_i(x) \in \mathcal{L}(\mathbb{K})$, and consider

$$\mathcal{L}(\Gamma) = \mathcal{L}(\mathbb{K}) + \{ \sharp_\gamma \mid \gamma \in \Gamma \}.$$

Intended meaning:

$$\sharp_\gamma \equiv LFP \text{ of } \gamma(x).$$

Example. $\mathbb{CTL}$ is a flat fixpoint logic :

$$\Gamma = \{ q \vee (p \wedge \Box x) \}, \qquad \sharp_{q \vee (p \wedge \Box x)} = p \mathcal{U} q.$$

Axiomatization issues

Problem

Find a complete axiomatization of $\mathcal{L}(\Gamma)$ *w.r.t.*
the standard semantics of Kripke frames (transition systems).

Uniform : find an algorithm such that, given Γ as input, produces as output a complete axiom system for $\mathcal{L}(\Gamma)$.

Axiomatization issues

Problem

Find a complete uniform axiomatization of the $\mathcal{L}(\Gamma)$s w.r.t. the standard semantics of Kripke frames (transition systems).

Uniform : find an algorithm such that, given Γ as input, produces as output a complete axiom system for $\mathcal{L}(\Gamma)$.

The usual axiomatization (Park and Kozen)

Axioms for modal logic $\mathbb{K}$, plus

$$\vdash \gamma(\sharp_\gamma) \to \sharp_\gamma\,, \qquad (\sharp_\gamma\text{-prefix})$$

$$\text{from } \vdash \gamma(\phi) \to \phi \text{ infer } \vdash \sharp_\gamma \to \phi\,. \qquad (\sharp_\gamma\text{-least})$$

Problem

Is the usual axiomatization complete w.r.t. Kripke frames?

- For many flat fixpoint logics yes, e.g. $\mathbb{CTL}$.
- For many others, we don't know, but ...

...we propose another axiomatization ...

...and prove it is sound and complete.

The problem upside down: flat fixpoint logics – as fragments of the modal μ-calculus

The language of $\mathcal{L}_\mu$:

$$
\begin{aligned}
\phi = x \mid \neg x \mid \top \mid \phi \wedge \phi \mid \bot \mid \phi \vee \phi \\
\mid \Diamond\phi \mid \Box\phi \\
\mid \mu x.\phi \mid \nu x.\phi
\end{aligned}
\qquad (x \text{ positive in } \phi.)
$$

Remark:

$$\mathbb{CTL} \subseteq \mathcal{L}_\mu\,, \qquad \mathcal{L}(\Gamma) \subseteq \mathcal{L}_\mu\,.$$

Theorem (Walukiewicz – and Kozen)

The usual axiomatization of $\mathcal{L}_\mu$ is complete w.r.t. Kripke frames.

Problem

Fragments of $\mathcal{L}_\mu$ for which the usual axiomatization is complete?
Alternative axiomatizations of fragments ensuring completeness?

Outline

Modal ♯-algebras

Def.

- *Modal algebra* (i.e. algebraic model of modal logic $\mathbb{K}$):

$$\mathfrak{A} = \langle A, \bot^{\mathfrak{A}}, \vee^{\mathfrak{A}}, \neg^{\mathfrak{A}}, \Diamond^{\mathfrak{A}} \rangle$$

  where $\langle A, \bot^{\mathfrak{A}}, \vee^{\mathfrak{A}}, \neg^{\mathfrak{A}} \rangle$ is a Boolean algebra, and $\Diamond^{\mathfrak{A}}$ is normal:

$$\Diamond^{\mathfrak{A}}(\bot^{\mathfrak{A}}) = \bot^{\mathfrak{A}}, \;\; \Diamond^{\mathfrak{A}}(x \vee^{\mathfrak{A}} y) = \Diamond^{\mathfrak{A}}(x) \vee^{\mathfrak{A}} \Diamond^{\mathfrak{A}}(y)\,.$$

- *Modal ♯-algebra*:

$$\mathfrak{A} = \langle A, \bot^{\mathfrak{A}}, \vee^{\mathfrak{A}}, \neg^{\mathfrak{A}}, \Diamond^{\mathfrak{A}}, \{\, \sharp_\gamma^{\mathfrak{A}} \mid \gamma \in \Gamma \,\} \rangle$$

  where $\langle A, \bot^{\mathfrak{A}}, \vee^{\mathfrak{A}}, \neg^{\mathfrak{A}}, \Diamond^{\mathfrak{A}} \rangle$ is a modal algebra, and

$$\sharp_\gamma^{\mathfrak{A}} \text{ is the } \mathit{LpFP}\text{of}\gamma^{\mathfrak{A}}\,, \;\text{ for all } \gamma \in \Gamma\,.$$

Kripke modal $\sharp$-algebras

If $\mathcal{M} = \langle S, \rightarrow \rangle$ is a Kripke frame, then

$$\mathfrak{M} = \langle \mathcal{P}(S), \emptyset, \cup, \overline{(\cdot)}, \Diamond \rangle$$

where

$$\Diamond(X) = \{\, s \in S \mid \exists s' \in X \text{ s.t. } s \rightarrow s' \,\}$$

is a modal algebra.

It is also a modal $\sharp$-algebra:
$\mathcal{P}(S)$ is a complete lattice hence, by Tarski-Knaster,
the LpFP of $\gamma^{\mathfrak{M}}$ exists and is uniquely determined.

Limits of Stone type completeness theorems

Proposition

Every Boolean algebra embeds into the powerset algebra of its ultrafilters.

Proposition

Every modal algebra embeds into the powerset algebra of some Kripke frame.

However:

Proposition (San08)

There is a modal $\sharp$-algebra that cannot be embedded into any complete modal $\sharp$-algebra.

The nonsense path to completeness

Each Kripke $\sharp$-algebra $\mathfrak{M} = \langle \mathcal{P}(S), \emptyset, \cup, \overline{(\cdot)}, \Diamond \rangle$ is

1. residuated,
2. constructive,
3. regular.

Proposition.

1. *The free modal $\sharp$-algebra is residuated and constructive.*
2. *The free regular modal $\sharp$-algebra is residuated and constructive.*

Proposition. *Each countable, residuated, constructive $\sharp$-algebra can be embedded into a Kripke $\sharp$-algebra.*

Corollary. *(Completeness) If a formula is valid in every Kripke frame, then it is also derivable in a formal system in which regularity is admissible.*

Outline

Residuation

Def. Let L, M be posets, $f : L \longrightarrow M$ be monotone.
f is *residuated* – or a *left adjoint* – if there exists $g : M \longrightarrow L$ such that

$$f(x) \leq y \quad \text{iff} \quad x \leq g(y)\,.$$

Example. In a Krikpe $\sharp$-algebra $\mathfrak{M} = \langle\, \mathcal{P}(S), \emptyset, \cup, \overline{(\,\cdot\,)}, \Diamond^{\mathfrak{M}} \,\rangle$ the monotone operation

$$f(X) = \Diamond^{\mathfrak{M}}(X)$$

is residuated because of

$$g(Y) := \{\, s \in S \mid \forall s' \;\; s' \to s \;\text{ implies }\; s' \in Y \,\}\,.$$

Def. A modal $\sharp$-algebra $\mathfrak{A}$ is *residuated* if $\Diamond^{\mathfrak{A}}$ is residuated.

Constructiveness

Def. We say that the LpFP $\mu.f$ of $f : L \longrightarrow L$ is *constructive* if

$$\mu.f = \bigvee_{\alpha \in Ord} f^{\alpha}(\bot)\,.$$

Def. A modal $\sharp$-algebra $\mathfrak{A}$ is *constructive* if

$$\sharp_{\gamma}^{\mathfrak{A}} = \bigvee_{\alpha \in Ord} (\gamma^{\mathfrak{A}})^{\alpha}(\bot)\,, \qquad \text{foreach } \gamma \in \Gamma\,.$$

Example. A Kripke $\sharp$-algebra is constructive.
Remark. Possible to construct $\sharp$-algebras that are persistently not contructive (San08).

From adjointness to $\mathcal{O}_{\mathfrak{f}}$-adjointness

Def. Let L, M be posets, $f : L \longrightarrow M$ be monotone.
f is a *left adjoint* if there exists $g : M \longrightarrow L$ such that

$$f(x) \leq y \quad \text{iff} \quad x \leq g(y)\,.$$

Def. $f : L \longrightarrow M$ is a (left) *$\mathcal{O}_{\mathfrak{f}}$-adjoint* if there exists $G : M \longrightarrow \mathcal{P}_{\mathfrak{f}}(L)$ such that

$$f(x) \leq y \quad \text{iff} \quad \exists z \in G(y) \text{ s.t. } x \leq z\,.$$

Remark. $f : L \longrightarrow M$ is an $\mathcal{O}_{\mathfrak{f}}$-adjoint iff

$$\mathcal{O}_{\mathfrak{f}}(f) : \mathcal{O}_{\mathfrak{f}}(L) \longrightarrow \mathcal{O}_{\mathfrak{f}}(M)$$

is a left adjoint – $\mathcal{O}_{\mathfrak{f}} : \mathrm{Pos} \longrightarrow \mathrm{Pos}$ free join-semilattice monad.

$\mathcal{O}_{\mathfrak{f}}$-adjoints and constructiveness

Proposition

If $f : L \longrightarrow L$ is a finitary $\mathcal{O}_{\mathfrak{f}}$-adjoint, then the LpFP of f, if it exists, is ω-constructive:

$$\mu.f = \bigvee_{n \geq 0} f^n(\bot)\,.$$

Def. An $\mathcal{O}_{\mathfrak{f}}$-adjoint $f : L \longrightarrow L$ is *finitary* if the set

$$\bigcup_{n \geq 0} G^n(x)$$

is finite, for each $x \in L$.

An example

Proposition

Let $\mathfrak{A}$ be a modal algebra and suppose that

1. $f : \mathfrak{A} \longrightarrow \mathfrak{A}$ *is a finitary left adjoint,*
2. *the least fixpoint*

$$f^*(y) = \mu_x.(y \vee f(x))$$

exists in $\mathfrak{A}$.

Then the relation

$$f^*(y) = \bigvee_{n \geq 0} f^n(y)$$

holds.

Proof of the Proposition

If $f^n(y) \leq z$, $n \geq 0$, then $y \leq g^n(z)$, $n \geq 0$, and

$$y \leq \bigwedge_{n \geq 0} g^n(z)\,. \tag{i}$$

Also:

$$\begin{aligned} f(\bigwedge_{n\geq 0} g^n(z)) &\leq \bigwedge_{n\geq 0} f(g^n(z)) \\ &= f(z) \wedge \bigwedge_{n\geq 0} f(g^{n+1}(z)) \\ &\leq f(z) \wedge \bigwedge_{n\geq 0} g^n(z) \leq \bigwedge_{n\geq 0} g^n(z) \end{aligned} \tag{ii}$$

Relations (i) and (ii) imply:

$$f^*(y) \leq \bigwedge_{n\geq 0} g^n(z) \leq z\,.$$

The free modal $\sharp$-algebra $\mathfrak{F}$

Def. $\mathfrak{F}$: Lindenbaum algebra of the logic $\mathcal{L}(\Gamma)$:

- elements are equivalence classes of formulas,
- $\phi \equiv \psi$ if $\vdash \phi \leftrightarrow \psi$
  in a standard formal system for $\mathbb{K}$
  augmented with Park's induction rules for the $\sharp_\gamma$, $\gamma \in \Gamma$.

Equivalently :
$\mathfrak{F}$ is the $\sharp$-modal algebra freely generated by a countable number of generators (propositional letters).

The (coalgebraic) cover modality on $\mathfrak{F}$

Let

$$\nabla X := \bigwedge_{x \in X} \Diamond x \wedge \Box \bigvee X \,.$$

Proposition

If Λ is a finite set of literals, then following condition holds in $\mathfrak{F}$:

$$\bigwedge \Lambda \wedge \nabla^{\mathfrak{F}} X \leq \bot^{\mathfrak{F}}$$

implies

$$\exists p \in \Lambda \text{ s.t. } \neg p \in \Lambda \quad \text{or} \quad \exists x \in X \text{ s.t. } x \leq \bot^{\mathfrak{F}} \,.$$

Corollary. *The operation*

$$\nabla^{\mathfrak{F}} : \mathcal{P}_{\mathfrak{f}}(\mathfrak{F}) \longrightarrow \mathfrak{F}$$

is an $\mathcal{O}_{\mathfrak{f}}$-adjoint on the free modal $\sharp$-algebra $\mathfrak{F}$.

The subformula property property for $\mathcal{L}(\Gamma)$

The previous Proposition-Corollary are equivalent to:

Proposition

Let Λ be a finite set of literals and let ψ, ϕ_i modal formulas, $i = 1, \ldots, n$. The following sound deductive rules:

$$\frac{\Lambda \vdash}{\Lambda, \Diamond\phi_1, \ldots, \Diamond\phi_n, \Box\psi \vdash} \; W_\Lambda$$

$$\frac{\phi_i, \psi \vdash}{\Lambda, \Diamond\phi_1, \ldots, \Diamond\phi_n, \Box\psi \vdash} \; L_i\Diamond + W$$

are jointly reversible.

Collecting results

Props.

1. *The operations*

$$\cdot \wedge^{\mathfrak{A}} k : \mathfrak{A} \longrightarrow \mathfrak{A}\,, \qquad \vee^{\mathfrak{A}} : \mathfrak{A} \times \mathfrak{A} \longrightarrow \mathfrak{A}\,,$$

   are $\mathcal{O}_{\mathfrak{f}}$-adjoints, for any modal $\sharp$-algebra $\mathfrak{A}$ and $k \in \mathfrak{A}$.

2. *The operation*

$$\nabla^{\mathfrak{F}} : \mathcal{P}_{\mathfrak{f}}(\mathfrak{F}) \longrightarrow \mathfrak{F}$$

   is a finitary $\mathcal{O}_{\mathfrak{f}}$-adjoint on $\mathfrak{F}$.

3. *All the operations in*

$$\mathrm{Clone}(\ \cdot \wedge k\ ,\ \vee\ ,\ \nabla\ ),$$

   – *thus $\Diamond, \Box$, and the disjunctive modal formulas,*

   are finitary $\mathcal{O}_{\mathfrak{f}}$-adjoints on $\mathfrak{F}$ . . .

4. *. . . whence their LpFPs are constructive.*

$\mathcal{O}_\mathfrak{f}$-ajointness and last rules

Let ϕ be a disjunctive formula.

There are jointly reversible left introduction rules for ϕ :

$$\frac{x \vdash \alpha_1}{\phi(x) \vdash \alpha} \quad \frac{x \vdash \alpha_2}{\phi(x) \vdash \alpha} \quad \ldots \quad \frac{x \vdash \alpha_n}{\phi(x) \vdash \alpha}$$

where α_i are computable subformulas of α and ϕ (not involving x).

$\mathcal{O}_{\mathfrak{f}}$-ajointness and cyclic proofs

Let ϕ be a disjunctive formula.

A cyclic proof of $\mu_x.\phi(x) \vdash \alpha_0$ has the form

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{x \vdash \alpha_0}{\phi(x) \vdash \alpha_2}}{x \vdash \alpha_2}}{\phi(x) \vdash \alpha_1}}{x \vdash \alpha_1}}{\phi(x) \vdash \alpha_0}}{x \vdash \alpha_0}$$

We cut it as :

$$\frac{x \vdash \alpha_1}{\phi(x) \vdash \alpha_0} \qquad \frac{x \vdash \alpha_2}{\phi(x) \vdash \alpha_1} \qquad \frac{x \vdash \alpha_0}{\phi(x) \vdash \alpha_2}$$

It follows :

$$\phi(\bigwedge_{i<3} \alpha_i) \vdash \bigwedge_{i<3} \alpha_i$$

$$\mu_x.\phi(x) \vdash \bigwedge_{i<3} \alpha_i \vdash \alpha_0 \,.$$

Summary : completeness for disjunctive Γs

If γ is a disjunctive formula – for all $\gamma \in \Gamma$ – then:

1. each $\gamma \in \Gamma$ is an $\mathcal{O}_{\mathfrak{f}}$-adjoint and has a constructive fixed point,
2. $\mathfrak{F}$ is countable, residuated, and constructive :
   it can embedded into the powerset algebra of some Kripke frame,
3. completeness of the standard axiomatization for $\mathcal{L}(\Gamma)$ follows.

Outline

Dealing with conjunctions: the subset construction

Lemma

Given $\{ \gamma_x \in \mathcal{L}(\mathbb{K}) \}_{x \in X}$, *there exists* $\{ \delta_Y \in \mathcal{L}(\mathbb{K}) \}_{Y \subseteq X, Y \neq \emptyset}$ *such that*

- δ_y *is in disjunctive normal form, i.e. constructed out of* $\vee$ *and special conjuctions:*

$$\bigwedge \Lambda \wedge \nabla X \,, \qquad\qquad \Lambda \text{ a finite set of literals}\,,$$

- *the relation*

$$\bigwedge_{x \in Y} \gamma_x = \delta_Y[\bigwedge Z/Z]$$

holds in every modal algebra.

Algebraic interpretation of the Lemma

Let

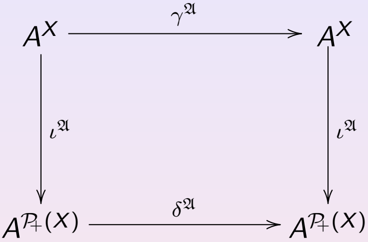
$$\mathcal{P}_{+}(X) = \{\, Y \subseteq X \mid Y \neq \emptyset \,\}\,, \qquad \iota_Y(\vec{v}) = \bigwedge_{x \in Y} \vec{v}_x\,.$$

On every modal algebra $\mathfrak{A}$:

$$\begin{array}{ccc} A^X & \xrightarrow{\;\gamma^{\mathfrak{A}}\;} & A^X \\ \downarrow{\scriptstyle \iota^{\mathfrak{A}}} & & \downarrow{\scriptstyle \iota^{\mathfrak{A}}} \\ A^{\mathcal{P}_{+}(X)} & \xrightarrow{\;\exists \delta^{\mathfrak{A}}\;} & A^{\mathcal{P}_{+}(X)} \end{array} \qquad (1)$$

commutes, with $\delta^{\mathfrak{A}}$ a finitary $\mathcal{O}_{\mathfrak{f}}$-adjoint (for the free algebra).

Dealing with conjunctions: the Transfer Lemma



Lemma (see Arnold & Niwinski)

If $\mathfrak{A}$ is a Kripke $\sharp$-algebra, then

$$\mu.\delta^{\mathfrak{A}} = \iota^{\mathfrak{A}}(\mu.\gamma^{\mathfrak{A}})\,. \qquad \text{(REG)}$$

Dealing with conjunctions: the Transfer Lemma

$$
\begin{array}{ccc}
A^X & \xrightarrow{\gamma^{\mathfrak{A}}} & A^X \\
\downarrow{\scriptstyle \iota^{\mathfrak{A}}} & & \downarrow{\scriptstyle \iota^{\mathfrak{A}}} \\
A^{\mathcal{P}_+(X)} & \xrightarrow{\delta^{\mathfrak{A}}} & A^{\mathcal{P}_+(X)}
\end{array}
$$

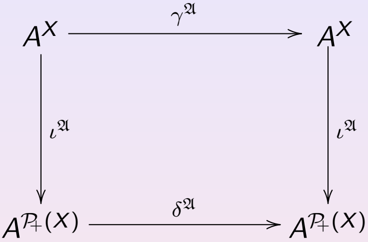
Proposition

If the least fixed point $\mu.\delta^{\mathfrak{A}}$ exists and is constructive, then the least fixed point $\mu.\gamma^{\mathfrak{A}}$ is also constructive, and

$$\mu.\delta^{\mathfrak{A}} = \iota^{\mathfrak{A}}(\mu.\gamma^{\mathfrak{A}}) \qquad \text{(REG)}$$

holds.

Dealing with conjunctions: the Transfer Lemma



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$$\mu.\delta^{\mathfrak{A}} = \iota^{\mathfrak{A}}(\mu.\gamma^{\mathfrak{A}}) \qquad \text{(REG)}$$

holds.

Regularity

Def. A modal $\sharp$-algebra $\mathfrak{A}$ is *regular* if

$$\iota^{\mathfrak{A}}(\mu.\gamma^{\mathfrak{A}}) \text{ is the least prefixed point of } \delta^{\mathfrak{A}}.$$

Proposition

If $\mathfrak{A}$ is regular, then

1. *the least fixed point of $\delta^{\mathfrak{A}}$ exists,*
2. *the identity*

$$\mu.\delta^{\mathfrak{A}} = \iota^{\mathfrak{A}}(\mu.\gamma^{\mathfrak{A}}) \qquad \text{(REG)}$$

   holds,

3. *$\mu.\gamma^{\mathfrak{A}}$ is constructive.*

The axiomatization

We axiomatize regularity !!!

For each $\gamma \in \Gamma$, do the following:

- Out of γ, construct a system of equations $\{ \gamma_x \mid x \in X \}$, whose least solution is equivalent to the existence of the LpFP of γ.
- Construct $\{ \delta_Y \mid Y \in \mathcal{P}_+(X) \}$ so to satisfy the Lemma.
- By adding inference rules (i.e. Horn clauses), impose the Lindenbaum algebra to be regular, i.e. impose that

$$\iota(\mu.\gamma) \text{ is the LpFP of } \delta\,.$$

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Conclusions, open problems

- Strength of algebraic interpretation of Walukiewicz proof tested on flat fragments.

TODOs :

- Climb the hierarchy.
- Is regularity derivable ? I.e. how much automata theoretic machinery do we need in the proof of completeness ?

Thank you !!!