Gentzen systems for Moss' coalgebraic logics

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Moss' coalgebraic modal logic

- ... is an expressive logic of Set-coalgebras, available for a large class of functors (functors preserving weak pullbacks) ...
  T-coalgebra

  $$\mathbb{C} : (X, c : X \to TX)$$

  ... based on a single cover modality, whose syntax and semantics is given automatically by the functor T:

  - T is the "arity" of the cover modality (syntax), and
  - relation lifting (of the $\Vdash$ relation) given by T provides semantics of the cover modality.

How far can we get if we want to do "classical" Gentzen-style proof theory for such language?

We can in fact define decent proof systems for such languages. Moreover, we do so uniformly for all the functors preserving weak pullbacks.

decent proof systems = sound, complete, cut-free analytic sequent systems.

Functors

- Power set functor P (covariant). P coalgebras = Kripke frames.
- Kripke polynomial functors:

$$T ::= Id \mid PT \mid T \times T \mid T + T \mid A$$

  Various automata can be given as coalgebras this way.
- Multiset functor N.

$$NX = \{f \mid f : X \to \mathbb{N}\}$$

- Distribution functor D.

$$DX = \{f \mid f : X \to [0,1], \sum_{x \in X} f(x) = 1\}$$

We mostly consider finitary versions of the functors, to obtain a finitary language.

Notation

- We consider coalgebras for (a finitary) endofunctor T_ω in Set
- We keep the notation:

$\mathcal{L}$	$a, b, c, \ldots$
$T_\omega \mathcal{L}$	$\alpha, \beta, \gamma \ldots$
$P_\omega \mathcal{L}$	$\varphi, \psi, \theta \ldots$
$T_\omega P_\omega \mathcal{L}$	$\Phi, \Psi, \Theta \ldots$
$P_\omega T_\omega \mathcal{L}$	$A, B, C \ldots$

Language

Modal language $\mathcal{L}$:

$$a ::= p \mid \neg p \mid \bigvee \varphi \mid \bigwedge \varphi \mid \nabla \alpha \mid \Delta \alpha$$

- If $\varphi \in P_\omega \mathcal{L}$ then $\bigwedge \varphi, \bigvee \varphi \in \mathcal{L}$
- If $\alpha \in T_\omega \mathcal{L}$ then $\nabla \alpha, \Delta \alpha \in \mathcal{L}$
- If $a \in \mathcal{L}$ then $\neg a \in \mathcal{L}$

Base: $T_\omega \to P_\omega$

$$\begin{aligned} Base(\alpha) &= \bigcap\{X \subseteq_\omega \mathcal{L} \mid \alpha \in T_\omega(X)\} \\ Base(\Phi) &= \bigcap\{\Psi \subseteq_\omega P_\omega\mathcal{L} \mid \Phi \in T_\omega(\Psi)\} \end{aligned}$$

for $A \in PT\mathcal{L}$ we use as its “base“ $\bigcup\{Base(\alpha) \mid \alpha \in A\}$.

Now subformulas of $\nabla\alpha$ can be defined as

$$Sub(\nabla\alpha) = Base(\alpha) \cup \{\nabla\alpha\}.$$

Semantics of nabla

Given a coalgebra $\mathbb{C}$ $(X, c : X \mapsto T_\omega X)$

$$X \ni s \Vdash \nabla\alpha \in \mathcal{L}$$

iff

$$T_\omega X \ni \; c(s) \; ??? \; \alpha \; \in T_\omega \mathcal{L}$$

Relation Lifting

If

$$T : Set \to Set$$

preserves weak pullbacks, it can be lifted to a functor

$$\overline{T} : Rel \to Rel$$

Relation Lifting

$$TX_1 \xleftarrow{T\pi_0} TR \xrightarrow{T\pi_1} TX_2$$

we define a lifted relation:

$$\overline{T}(R) = \{((T\pi_0)w, (T\pi_1)w \mid w \in TR\}$$

Relation Lifting

$$\alpha \overline{T}(R)\beta \quad \text{iff} \quad \exists w \in TR \qquad \begin{array}{l} (T\pi_0)w = \alpha \\ (T\pi_1)w = \beta \end{array} \tag{1}$$

Relation Lifting

Given $R \subseteq X_1 \times X_2$, the lifting of R is $\overline{P}(R) \subseteq PX_1 \times PX_2$ given by

$$\overline{P}(R) = \{(U, V) \mid \begin{array}{ll} \forall u \in U & \exists v \in V\, uRv \\ \forall v \in V & \exists u \in U\, uRv \} \end{array} \tag{1}$$

Semantics of nabla

Given a coalgebra $\mathbb{C}$ $(X, c : X \mapsto T_\omega X)$

$$\mathbb{C}, s \Vdash \nabla\alpha$$

iff

$$c(s)\overline{T}(\Vdash)\alpha$$

Semantics of delta

Given a coalgebra $\mathbb{C}$ $(X, c : X \mapsto T_\omega X)$

$$\mathbb{C}, s \not\Vdash \Delta\alpha$$

iff

$$c(s)\overline{T}(\not\Vdash)\alpha$$

Semantics of nabla - P_ω

Given a coalgebra $\mathbb{C}$ $(X, c : X \mapsto P_\omega X)$

$$\mathbb{C}, s \Vdash \nabla\alpha$$

iff

$$(\forall t \in c(s))(\exists a \in \alpha)\ t \Vdash a$$

and

$$(\forall a \in \alpha)(\exists t \in c(s))\ t \Vdash a$$

meaning that α is distributed over $c(s)$ so that $c(s)$ and (values of formulas from) α cover each other.

Redistributions

Suppose $T = P$ and

$$s \Vdash \bigwedge\{\nabla\alpha \mid \alpha \in A\}$$

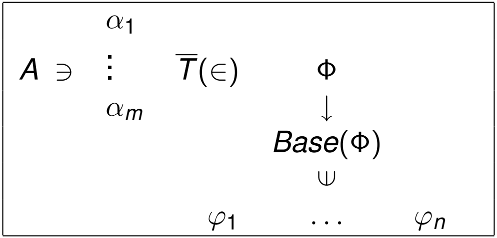
consider, for a $t \in c(s)$, a set

$$\{a \in \bigcup A \mid t \Vdash a\},$$

we can say A is redistributed over $c(s)$ to the set

$$\Phi = \{\{a \in \bigcup A \mid t \Vdash a\} \mid t \in c(s)\}.$$

Redistributions



Φ is a slim redistribution of A iff

- $\Phi \in TP \bigcup_{\alpha \in A} Base(\alpha)$
- $A \subseteq \{\alpha \mid \alpha \overline{T}(\in) \Phi\}$

P_ω redistributions - examples

A classical distributive law:

$$\bigvee\{\bigwedge \alpha \mid \alpha \in A\} \equiv \bigwedge\{\bigvee \varphi \mid \varphi \overline{P}(\in) A\}$$

where

$$\{\varphi \mid \varphi \overline{P}(\in) A\} \in SRD(A)$$

P_ω redistributions - examples

$$s \Vdash \bigwedge\{\nabla\alpha \mid \alpha \in A\} \iff \{\{a \in \bigcup A \mid t \Vdash a\} \mid t \in c(s)\} \in SRD(A).$$

P_ω redistributions - examples

$A = \varnothing$ has only two slim redistributions:

$$SRD(\varnothing) = \{\varnothing, \{\varnothing\}\}.$$

Basic calculus $G\nabla\Delta$

Sequents of the form $\varphi \Rightarrow \psi$, the usual reading $\bigwedge \varphi \to \bigvee \psi$

Basic calculus $G\nabla\Delta$

Axioms and boolean rules

$$a \Rightarrow a$$

$$\bigwedge\text{-l}\ \frac{\varphi, \theta \Rightarrow \psi}{\varphi, \bigwedge\theta \Rightarrow \psi} \qquad \bigvee\text{-r}\ \frac{\varphi \Rightarrow \theta, \psi}{\varphi \Rightarrow \bigvee\theta, \psi}$$

$$\bigwedge\text{-r}\ \frac{\{\varphi \Rightarrow a, \psi \mid a \in \theta\}}{\varphi \Rightarrow \bigwedge\theta, \psi} \qquad \bigvee\text{-l}\ \frac{\{\varphi, a \Rightarrow \psi \mid a \in \theta\}}{\varphi, \bigvee\theta \Rightarrow \psi}$$

Basic calculus $G\nabla\Delta$

$$\neg\text{-r}\ \frac{\varphi, a \Rightarrow \psi}{\varphi \Rightarrow \psi, \neg a} \qquad \neg\text{-l}\ \frac{\varphi \Rightarrow a, \psi}{\varphi, \neg a \Rightarrow \psi}$$

$$\text{w-l}\ \frac{\varphi \Rightarrow \psi}{\varphi, \varphi' \Rightarrow \psi} \qquad \text{w-r}\ \frac{\varphi \Rightarrow \psi}{\varphi \Rightarrow \psi, \psi'}$$

Basic calculus $G\nabla\Delta$

... and modal rules

One-sided variant $G\nabla$, ∇ only

sequents of the form $\varphi \Rightarrow \varnothing$, reading '$\bigwedge \varphi$ is not satisfiable'

One-sided variant $G\nabla$, ∇ only

$$p, \neg p \Rightarrow \varnothing$$

$$\bigwedge\text{-l}\ \frac{\varphi, \psi \Rightarrow \varnothing}{\varphi, \bigwedge \psi \Rightarrow \varnothing} \qquad \bigvee\text{-l}\ \frac{\{\varphi, a \Rightarrow \varnothing | a \in \psi\}}{\varphi, \bigvee \psi \Rightarrow \varnothing} \qquad \text{weak-l}\ \frac{\varphi \Rightarrow \varnothing}{\varphi, \varphi' \Rightarrow \varnothing}$$

One-sided variant $G\nabla$, ∇ only

... and a ∇ rule ...

One-sided variant $G\nabla$, ∇ only

$$\nabla\text{-1} \; \frac{\{\varphi_\Phi \Rightarrow \varnothing | \Phi \in \mathit{SRD}(A)\}}{\{\nabla\alpha | \alpha \in A\} \Rightarrow \varnothing} \; \forall\Phi.\varphi_\Phi \in \mathit{Base}(\Phi)$$

... to be read as follows: Given A, if for every $\Phi \in \mathit{SRD}(A)$ there exists some $\varphi_\Phi \in \mathit{Base}(\Phi)$ such that $\varphi_\Phi \Rightarrow \varnothing$, then $\{\nabla\alpha | \alpha \in A\} \Rightarrow \varnothing$.

$G\nabla$ – soundness, completeness

A distributive law behind the rule:

$$\bigwedge\{\nabla\alpha \mid \alpha \in A\} \equiv \bigvee\{\nabla(T\bigwedge)\Phi \mid \Phi \in SRD(A)\}$$

$G\nabla$ – soundness, completeness

If $\{\nabla\alpha | \alpha \in A\}$ is satisfiable, then there is $\Phi \in SRD(A)$ with all $\varphi_\Phi \in Base(\Phi)$ satisfiable.

Suppose there is a coalgebra $\mathbb{C} = (X, c)$ and a state $s \Vdash \bigwedge\{\nabla\alpha | \alpha \in A\}$. Define

$$\varphi : X \to P(\bigcup_{\alpha \in A} Base(\alpha))$$

by

$$s \mapsto \{a \mid s \Vdash a\}.$$

Then put

$$\Phi \;=\; (T\varphi)c(s).$$

$G\nabla$ – soundness, completeness

If there is $\Phi \in \mathit{SRD}(A)$ with all $\varphi \in \mathit{Base}(\Phi)$ satisfiable, then $\{\nabla\alpha | \alpha \in A\}$ is satisfiable.

Suppose for each $\varphi \in \mathit{Base}(\Phi)$ there is a coalgebra $\mathbb{C}_\varphi = (X_\varphi, c_\varphi), s_\varphi \Vdash \bigwedge \varphi$. Then $s : \mathit{Base}(\Phi) \to \biguplus X_\varphi$. We form a disjoint union of them, adding one new state:

$$\begin{aligned} X &= \biguplus X_\varphi \uplus \{t\} \\ c(x) &= c_\varphi(x) \quad \text{for } x \in X_\varphi \\ c(t) &= (Ts)(\Phi) \end{aligned}$$

Now $t \Vdash \bigwedge\{\nabla\alpha | \alpha \in A\}$.

An example of a proof - P_ω

$$\nabla\text{-}1\ \frac{\{\varphi_\Phi \Rightarrow \varnothing \mid \Phi \in SRD(A)\}}{\{\nabla\alpha \mid \alpha \in A\} \Rightarrow \varnothing}\ \varphi_\Phi \in \Phi$$

$$\frac{\bot \Rightarrow \varnothing}{\nabla\{\bot\} \Rightarrow \varnothing}$$

where $A = \{\{\bot\}\}$, $\Phi = \{\{\bot\}\}$ is the only slim redistribution of A.

An example of a proof - P_ω

$$\nabla\text{-1}\ \frac{\{\varphi_\Phi \Rightarrow \varnothing | \Phi \in SRD(A)\}}{\{\nabla\alpha | \alpha \in A\} \Rightarrow \varnothing}\ \varphi_\Phi \in \Phi$$

Suppose α contains $\bot$.

$$\frac{\{\varphi_\Phi \Rightarrow \varnothing \mid \Phi \in SRD(\{\alpha\})\}}{\nabla\alpha \Rightarrow \varnothing}$$

where $A = \{\alpha\}$, and for each $\Phi \in SRD(\{\alpha\})$ some $\varphi \in \Phi$ must contain $\bot$.

∇ and Δ

In general, ∇ can be expressed using Δ, and vice versa:

- $\Delta\alpha \equiv \neg\nabla(T\neg)\alpha$
- $\nabla\alpha \equiv \neg\Delta(T\neg)\alpha$

∇ and Δ

They remain equi-expressible even without the negation being present:

- $\Delta\alpha \equiv \bigvee\{\nabla\beta \mid \beta \in L_T\alpha\}$
- $\nabla\alpha \equiv \bigwedge\{\Delta\beta \mid \beta \in R_T\alpha\}$

where we define

$$
\begin{aligned}
D(\alpha) &= \{\Phi \in TPBase(\alpha) \mid (\alpha, \Phi) \notin \overline{T}(\notin)\},\\
L_T(\alpha) &= \{(T\bigwedge)\Phi \mid \Phi \in D(\alpha)\},\\
R_T(\alpha) &= \{(T\bigvee)\Phi \mid \Phi \in D(\alpha)\}.
\end{aligned}
$$

∇ and Δ – P_ω

- $\Delta\alpha \equiv \bigvee\{\nabla\beta \mid \beta \in L_P\alpha\}$
- $\nabla\alpha \equiv \bigwedge\{\Delta\beta \mid \beta \in R_P\alpha\}$

$$L_P\alpha = \begin{cases} \{\{\varnothing\} \cup \{\{a\} \mid a \in \alpha\} \cup \{\{\bigwedge \alpha, \top\}\}\} & \text{if } \alpha \neq \varnothing \\ \{\{\top\}\} & \text{if } \alpha = \varnothing \end{cases} \tag{2}$$

$$R_P\alpha = \begin{cases} \{\{\varnothing\} \cup \{\{a\} \mid a \in \alpha\} \cup \{\{\bigvee \alpha, \bot\}\}\} & \text{if } \alpha \neq \varnothing \\ \{\{\bot\}\} & \text{if } \alpha = \varnothing \end{cases} \tag{3}$$

Basic calculus $G\nabla\Delta$ – modal rules

$$\Delta\text{-l}\ \frac{\{\varphi, \nabla\beta \Rightarrow \psi | \beta \in L\alpha\}}{\Delta\alpha, \varphi \Rightarrow \psi} \qquad \nabla\text{-r}\ \frac{\{\varphi \Rightarrow \Delta\beta, \psi | \beta \in R\alpha\}}{\varphi \Rightarrow \nabla\alpha, \psi}$$

- $\Delta\alpha \equiv \bigvee\{\nabla\beta \mid \beta \in L_P\alpha\}$
- $\nabla\alpha \equiv \bigwedge\{\Delta\beta \mid \beta \in R_P\alpha\}$

Basic calculus $G\nabla\Delta$ – modal rules

$$\nabla\Delta \; \frac{\{\varphi_A^\Phi \Rightarrow \varphi_B^\Phi \mid \Phi \in SRD(A \uplus B)\}}{\{\nabla\alpha \mid \alpha \in A\} \Rightarrow \{\Delta\beta \mid \beta \in B\}} \; \forall\Phi.\varphi^\Phi \in Base(\Phi)$$

Where $\varphi_A^\Phi = f_0^{-1}\varphi^\Phi$ and $\varphi_B^\Phi = f_1^{-1}\varphi^\Phi$, and $f_0 : A \to A \uplus B$ and $f_1 : B \to A \uplus B$ are the injection maps.

Example of a proof – *P*

$$\dfrac{\dfrac{\emptyset}{\Rightarrow \Delta\emptyset, \Delta\{\bot\}}\ \nabla\Delta}{\Rightarrow \Delta\emptyset, \nabla\emptyset}\ \nabla\text{-r}$$

Example of a proof – *P*

$$\nabla\Delta\ \frac{a \Rightarrow a \qquad a \Rightarrow a, \bot}{\nabla\{a\} \Rightarrow \Delta\{a, \bot\}}$$