Algebras, simulations, and provable ordinals

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Gentzen's 'trick'

Beweisbarkeit und Unbeweisbarkeit von Anfangsfällen der transfiniten Induktion in der reinen Zahlentheorie*).

Von

Gerhard Gentzen in Göttingen.

Section **2.2**, concerned with provability:

2. 2. *Umformung einer TJ-Herleitung bis* ω_n *in eine TJ-Herleitung bis* $\omega_{n+1} = \omega^{\omega_n}$. ($n$ bezeichne eine natürliche Zahl oder 0.)

General location of topic

The topic has to do with algebras, specifically initial algebras for certain non-finitary functors such as

$$\begin{array}{rcl} X \mapsto \mathbb{1} + X + (\mathbb{N} \to X) & : & \mathsf{Set} \to \mathsf{Set} \\ P \mapsto \{a : O \mid seg\ a \subseteq P\} & : & \mathbb{P}\, O \to \mathbb{P}\, O\ . \end{array}$$

In the indexed version, O is an ordered set (of ordinal notations), $\mathbb{P}\, O$ is a type of predicates or set-valued functions on O, and seg_a is a cofinal family of immediate predecessors of a.
A lens is a transformer of algebras for such functors. It implements an arithmetic function at the level of ordinals, typically by means of an operation at the level of types.

(Im)Predicative arithmetic

Suppose $\mathbb{N} \triangleq \Pi_X (X \to X) \to X \to X$ is a possible value of X.
$\mathbb{N} \cong \Pi_X (F\,X \to X) \to X$ where $F\,X = X + \mathbb{1}$.

$$\begin{array}{lcl} 0_X\,(s, z) & = & z \\ (Suc\,n)_X\,(s, z) & = & s\,(n_X\,(s, z)) \\ m + n & = & n_{\mathbb{N}}\,(Suc, m) \\ 2 \,\hat{}\, n & = & n_{\mathbb{N}}\,(m \mapsto m + m)\,(Suc\,0) \end{array}$$

Suppose not.

$$\begin{array}{lcl} (m+n)_X\,(s, z) & = & n_X\,(s, m_X\,(s, z)) \\ (2 \,\hat{}\, n)_X\,(s, z) & = & \underbrace{\pi_z}_{\text{'extractor'}} \; n_{\underbrace{X \to X}_{\text{carrier}}} \underbrace{(f \mapsto f \cdot f, s)}_{\text{algebra}} \end{array}$$

L,U,D

If $m_X, n_X : ((X+\mathbb{1}) \to X) \to X$,

$$\left.\begin{array}{lcll} 0_X & = & (\_, z) & \mapsto z \\ (m+n)_X & = & (s, z) & \mapsto n_X(s, m_X(s, z)) \\ (2^n)_X & = & a & \mapsto D_X\, a(n_{(L\,X)}(U_X\, a)) \end{array}\right\} : ((X+\mathbb{1}) \to X) \to X$$

where

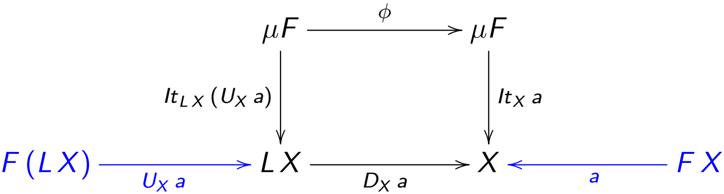
$$\begin{array}{lcllll} L & = & (X & :\text{Set} &) \mapsto X \to X & :\text{Set} \\ U_X & = & ((s, \_) & :(X+\mathbb{1}) \to X &) \mapsto (twice, s) & :(L\,X+\mathbb{1}) \to L\,X \\ D_X & = & ((\_, z) & :ditto &) \mapsto (f : L\,X \mapsto f\,z) & :L\,X \to X \end{array}$$

Simulation of ϕ by (L, U, D)

The category is Set, the endofunctor F is something like

$$X \mapsto 1 + X + (\mathbb{N} \to X) : \mathrm{Set} \to \mathrm{Set}$$

and $\phi : \mu F \to \mu F$ is something like $(2\,\hat{}\,)$, $(\omega\,\hat{}\,)$.



$$It_X a \cdot \phi = D_X\ a \cdot It_{L\,X}\,(U_X\ a)$$

Indexed version

The category is Set^O, where O is a transitive order. The endofunctor F is something like:

$$(U : O \to \mathrm{Set}) \mapsto \{a : O \,|\, seg\ a \subseteq U\}$$

where $seg\ a$ is *e.g.* a cofinal family of immediate predecessors of a. (Or the entire initial segment of O below a.)

The algebras of F are *progressive* predicates. An *accessible* element is the least progressive predicate. $Acc = \mu F$.

The function $\phi : O \to O$ is a symbolic function such as $(2\,\hat{}\,)$, $(\omega\,\hat{}\,)$, or a section of the 2-place Veblen function over one of these, and $\tilde{\phi}$ is a proof that the accessible part of O is closed under ϕ, *i.e.* $\exists_\phi Acc \to Acc$.

$$\begin{array}{ccc}
\exists_\phi Acc & \xrightarrow{\quad \tilde{\phi} \quad} & Acc \\
{\scriptstyle \exists_\phi(It_{(L\,X)}\,(U_X\ a))} \big\downarrow & & \big\downarrow {\scriptstyle It_X\ a} \\
\exists_\phi(L\,X) & \xrightarrow[\quad D_X\ a \quad]{} & X
\end{array}$$

Binary composition

Functions $\phi, \psi : \Omega \to \Omega$ that have lenses are closed under composition:

$$\begin{array}{rl} & \mathit{It} X\, a \cdot \phi \cdot \psi \\ = & D_\phi\, X\, a \cdot \mathit{It}(L_\phi\, X)\,(U_\phi\, X\, a) \cdot \psi \\ = & D_\phi\, X\, a \cdot D_\psi\,(L_\phi\, X)\,(U_\phi\, X\, a) \cdot \mathit{It}(L_\psi\,(L_\phi\, X))\,(U_\psi(L_\phi\, X)\,(U_\phi\, X\, a)) \end{array}$$

So

$$\begin{array}{rcl} L_{\phi\cdot\psi} & = & L_\psi \cdot L_\phi \\ U_{\phi\cdot\psi}\, X & = & U_\psi\,(L_\phi X) \cdot U_\phi\, X \\ D_{\phi\cdot\psi}\, X\, a & = & D_\phi\, X\, a \cdot (D_\psi\,(L_\phi\, X) \cdot U_\phi\, X)\, a \end{array}$$

Infinitary composition: the derivative

Suppose that for $n : \mathbb{N}$, $\phi_n : \Omega \to \Omega$ is normal (strictly increasing and continuous) with lens (L_n, U_n, D_n).
Let ϕ enumerate $\{a : \Omega \,|\, (\Pi\, n : \mathbb{N})\, a = \phi_n\, a\}$. (Veblen's *derivative*.)
We can define (using transfinite *types*) a lens (L, U, D) for ϕ.
Not at all tricky, but a bit too lengthy to explain here.

Lenses carry an algebra

Gentzen gave us a lens for $(\omega\,\hat{}\,)$. We have an operation taking a countable sequence of lenses to their derivative. So we have an algebra for the functor

$$X \;\mapsto\; 1 + X + (\mathbb{N} \to X)$$

The *carrier* is the (large) type

$$\begin{array}{rll} \mathit{Lens} = & (\Sigma\, L : \mathrm{Set} \to \mathrm{Set}) & \\ & & [(X : \mathrm{Set}) \to (F\,X \to X) \to F\,(L\,X) \to L\,X] \\ & \times & [(X : \mathrm{Set}) \to (F\,X \to X) \to L\,X \to X] \end{array}$$

The *structure map* on lenses combines

- zero case: the Gentzen lens.
- successor case: the (unary) derivative operation (infinitary composition of a constant sequence).
- limit case: derivative of a sequence, infinitary composition.

'Meta' lenses

The notion of 'lens' can be relativised to a universe of sets (U, T). We can use a 'meta'-lens (in the next universe) for $+\omega^\beta$ to generate a lens (in this universe) for ϕ_β.
This is a manifestation of what weirdly resembles an 'adjunction'

$$\begin{array}{ccc} \Gamma \vdash_{\gamma}^{\alpha+\omega^\beta} A & \Rightarrow & \Gamma \vdash_{\phi_\beta \gamma}^{\alpha} A \\ [\alpha + \omega^\beta, \gamma] & \cong & [\alpha, \phi_\beta \gamma] \\ (+\omega^\beta) & \dashv & \phi_\beta \end{array}$$

pervading sub-Γ_0 proof theory.

(Admittedly, this is more of a vivid hallucination than a precise conjecture.)

Summary, and confession

- It seems (to me) indubitable that there is a lot of algebraic structure lurking beneath the surface of well-ordering proofs (‘lower bounds’). The same can perhaps be said for ordinally informative cut-elimination proofs (‘upper bounds’).
- I don’t really know how to properly capture algebraic structure in categorical terms. My hope is to interest someone here more adept than I with categorical concepts and techniques.

Over to you.

Some details of infinitary composition

Given a sequence of lenses :

$L_n : \mathrm{Set} \to \mathrm{Set}$

$U_n : (X : \mathrm{Set}) \to (F\,X \to X) \to F\,(L_n\,X) \to L_n\,X$

$D_n : (X : \mathrm{Set}) \to (F\,X \to X) \to L_n\,X \to X$

$$
\begin{array}{ll}
\overline{L}_0 = id & \overline{L}_{n+1} = L_n \cdot \overline{L}_n \\
\overline{U}_0\,X = id & \overline{U}_{n+1}\,X = U_n\,(\overline{L}_n\,X) \cdot \overline{U}_n\,X \\
\overline{D}_0\,X\,\_ = id & \overline{D}_{n+1}\,X\,a = \overline{D}_n\,X\,a \cdot (D_n\,(\overline{L}_n\,X) \cdot \overline{U}_n\,X)\,a
\end{array}
$$

Let $L : \mathrm{Set} \to \mathrm{Set}$ be $X \mapsto \Pi_n(\overline{L}_n\,X)$. Fix $X : \mathrm{Set}, a : F\,X \to X$.

Let $\mathfrak{l} : (\mathbb{N} \to L\,X) \to L\,X$ be $\xi, n \mapsto \overline{U}_n(X, a)_{.\mathbf{lim}}\,(m \mapsto \xi(m, n)))$.

Let $\downarrow : L\,X \to L\,X$ be $\xi, n \mapsto D_n\,(\overline{L}_n\,X)(\overline{U}_n\,X\,a)(\xi\,(n+1))$.

Let $\downarrow^\omega : L\,X \to L\,X$ be $\xi \mapsto \mathfrak{l}(n \mapsto \downarrow^n \xi)$.

Let $\mathfrak{s} : L\,X \to L\,X$ be $\xi \mapsto \downarrow^\omega (n \mapsto \overline{U}_n(X, a)_{.\mathbf{succ}}(\xi\,n))$.

Let $\mathfrak{z} : L\,X$ be $\downarrow^\omega (n \mapsto \overline{U}_n(X, a)_{.\mathbf{zero}})$.

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Let $\downarrow^{\omega} : L\,X \to L\,X$ be $\xi \mapsto \mathfrak{l}(n \mapsto \downarrow^n \xi)$.
Let $\mathfrak{s} : L\,X \to L\,X$ be $\xi \mapsto \downarrow^{\omega}\,(n \mapsto \overline{U}_n(X, a)_{\mathbf{.succ}}(\xi\,n))$.
Let $\mathfrak{z} : L\,X$ be $\downarrow^{\omega}\,(n \mapsto \overline{U}_n(X, a)_{\mathbf{.zero}})$.
Then $U(X, a)$ is $(\mathfrak{z}, \mathfrak{s}, \mathfrak{l})$, and $D(X, a) = \xi \mapsto \xi\,0$.