

An Algebraic Proof of the Disjunction Property

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Outline

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- 2 Substructural logics
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- 4 Why is the DP interesting?
- 5 How to prove the DP?

Complexity issues

- Proving a complexity of the decision problem of a (substructural) logic is often a tedious task usually based on a detailed inspection of the underlying (sequent) calculus.
- Such results often rely on proof theoretic methods and presuppose that the logic under consideration possesses a good sequent calculus for which cut-elimination holds.
- A typical example is **MALL** which is known to be PSPACE-complete (Lincoln et al.). The proof is very long and technical.
- Can we have uniform methods which work for wider classes of substructural logics?

Base logic

- Our base logic is the **full Lambek calculus FL** (multiplicative-additive fragment of noncommutative intuitionistic linear logic).
- Multiplicative connectives: $\cdot, \backslash, /, 1, 0$,
- Additive connectives: $\vee, \wedge, \perp, \top$.
- **FL** is given by a single-conclusion sequent calculus:

$$\alpha \Rightarrow \alpha \quad \Rightarrow 1 \quad 0 \Rightarrow$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Pi, \alpha, \Sigma \Rightarrow \varphi}{\Pi, \Gamma, \Sigma \Rightarrow \varphi} \text{ (cut)}$$

$$\frac{\Gamma, \alpha, \Sigma \Rightarrow \varphi \quad \Gamma, \beta, \Sigma \Rightarrow \varphi}{\Gamma, \alpha \vee \beta, \Sigma \Rightarrow \varphi} (\vee \Rightarrow) \quad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi} (\Rightarrow \vee) \quad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \vee \psi} (\Rightarrow \vee)$$

$$\vdots$$

Substructural logics

Definition

A **substructural logic** is an extension of **FL** by a set of rules (axioms) closed under substitutions having the form:

$$\frac{\Gamma_1 \Rightarrow \varphi_1 \quad \cdots \quad \Gamma_n \Rightarrow \varphi_n}{\Gamma_0 \Rightarrow \varphi_0}$$

Example

$$\frac{\Gamma, \alpha, \alpha, \Delta \Rightarrow \varphi}{\Gamma, \alpha, \Delta \Rightarrow \varphi} \text{ (c)} \quad \frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \varphi}{\Gamma, \beta, \alpha, \Delta \Rightarrow \varphi} \text{ (e)} \quad \frac{\Gamma, \Delta \Rightarrow \varphi}{\Gamma, \alpha, \Delta \Rightarrow \varphi} \text{ (i)} \quad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \alpha} \text{ (o)}$$

- **MALL** = **InFL**_e = **FL**+(e)+(α \ 0) \ 0 ⇒ α,
- **Int** = **FL**+(e)+(c)+(i)+(o).

Algebraic semantics

Definition

An **FL-algebra** is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, /, \backslash, 0, 1 \rangle$, where

- $\langle A, \wedge, \vee \rangle$ is a lattice,
- $\langle A, \cdot, 1 \rangle$ is a monoid,
- 0 is an arbitrary element and
- the following condition holds:

$$x \cdot y \leq z \quad \text{iff} \quad x \leq z/y \quad \text{iff} \quad y \leq x \backslash z.$$

Fact

The class of FL-algebras form a variety (i.e., an equational class).

Algebraizability

- **FL** is algebraizable and its equivalent algebraic semantics is the variety of **FL-algebras**.
- Thus there is a dual isomorphism **Q** between the lattice of substructural logics and the sub-quasivariety lattice of FL-algebras.
- Let **L** be a substructural logic. Then we have the following equivalences:

$$\vdash_{\mathbf{L}} \varphi \quad \text{iff} \quad \models_{\mathbf{Q}(\mathbf{L})} 1 = 1 \wedge \varphi \quad [1 \leq \varphi].$$

$$\models_{\mathbf{Q}(\mathbf{L})} \varphi = \psi \quad \text{iff} \quad \vdash_{\mathbf{L}} (\varphi \setminus \psi) \wedge (\psi \setminus \varphi).$$

- By **complexity** of a substructural logic **L** we mean the complexity of its set of theorems. Due to algebraizability it is the same as the complexity of the **equational theory** for $\mathbf{Q}(\mathbf{L})$.

Correspondence between logic and algebra

Logic	Algebra
logic FL axiom φ inference rule (r) axiomatic extension L of FL rule extension L of FL consistent	variety FL identity $1 \leq \varphi$ quasi-identity (r) subvariety $V(\mathbf{L})$ of FL subquasivariety $Q(\mathbf{L})$ of FL nontrivial

- A logic **L** is **consistent** if there φ such that $\not\vdash_{\mathbf{L}} \varphi$.

Disjunction Property

Definition

Let \mathbf{L} be a substructural logic. Then \mathbf{L} satisfies the **disjunction property** (DP) if for all formulas φ, ψ

$$\vdash_{\mathbf{L}} \varphi \vee \psi \quad \text{implies} \quad \vdash_{\mathbf{L}} \varphi \quad \text{or} \quad \vdash_{\mathbf{L}} \psi.$$

Analogously, we say the a quasivariety \mathbf{K} of FL-algebras has the DP if

$$\models_{\mathbf{K}} 1 \leq \varphi \vee \psi \quad \text{implies} \quad \models_{\mathbf{K}} 1 \leq \varphi \quad \text{or} \quad \models_{\mathbf{K}} 1 \leq \psi.$$

DP and complexity

Theorem (Chagrov, Zakharyashev)

Every consistent superintuitionistic logic having the DP is PSPACE-hard.

Theorem

Every consistent substructural logic having the DP is PSPACE-hard.

Proof.

By reduction from the set of true quantified Boolean formulas. □

Remark

One cannot use the coding of quantifiers from MALL. It does not work for some logics having the DP, e.g. $\mathbf{FL} + \alpha\beta \wedge \alpha\gamma \Rightarrow \alpha(\beta \wedge \gamma)$.

Proof-theoretic proof of DP

- How to prove the DP?
- If our logic \mathbf{L} enjoys the **cut-elimination** then one can use the following.
- The very last rule in every **cut-free** proof of $\Rightarrow \varphi \vee \psi$ has to be $(\Rightarrow \vee)$. Thus either $\Rightarrow \varphi$ or $\Rightarrow \psi$ is provable.
- What can we do if our logic does not have a cut-free presentation?
E.g. if \mathbf{L} is the extension of \mathbf{FL} by $\alpha \setminus \alpha\beta \Rightarrow \beta$ and $\beta\alpha/\alpha \Rightarrow \beta$.

Algebraic characterization of the DP

Definition

An FL-algebra \mathbf{A} is called **well-connected** if for all $x, y \in A$, $x \vee y \geq 1$ implies $x \geq 1$ or $y \geq 1$.

Theorem

Let \mathbf{L} be a substructural logic. Then \mathbf{L} has the DP iff the following condition holds:

() for every $\mathbf{A} \in \mathbf{Q}(\mathbf{L})$ there is a well-connected FL-algebra $\mathbf{C} \in \mathbf{Q}(\mathbf{L})$ such that \mathbf{A} is a homomorphic image of \mathbf{C} .*

Using this theorem we would like to find a large class of quasi-varieties of FL-algebras having the DP.

ℓ -monoidal quasi-identities (rules)

Definition

An ℓ -monoidal quasi-identity is a quasi-identity

$$t_1 \leq u_1 \text{ and } \dots \text{ and } t_n \leq u_n \implies t_0 \leq u_0,$$

where t_i is in the language $\{\cdot, \wedge, \vee, 1\}$ and u_i is either 0 or in the language $\{\cdot, \wedge, \vee, 1\}$.

Accordingly, an ℓ -monoidal rule is a rule

$$\frac{\Gamma_1 \Rightarrow \varphi_1 \quad \dots \quad \Gamma_n \Rightarrow \varphi_n}{\Gamma_0 \Rightarrow \varphi_0}$$

where Γ_i is a sequence of formulas in the language $\{\cdot, \wedge, \vee, 1\}$ and φ_i is either empty or a formula in the language $\{\cdot, \wedge, \vee, 1\}$.

Useful algebra

- Fix a quasivariety K of FL-algebras defined by ℓ -monoidal quasi-identities.

Lemma

For any nontrivial algebra $\mathbf{A} \in K$, there is an integral FL-algebra $\mathbf{B} \in K$ which has a *unique subcover of 1*.

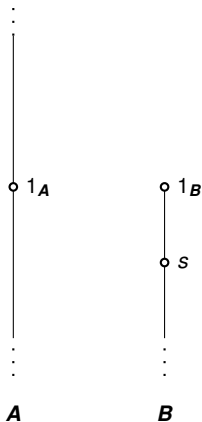
Proof.

Let $a \in A$ such that $a < 1$ and $B = \{a^n \mid n \geq 0\}$. The submonoid B gives rise to an FL-algebra \mathbf{B} by setting

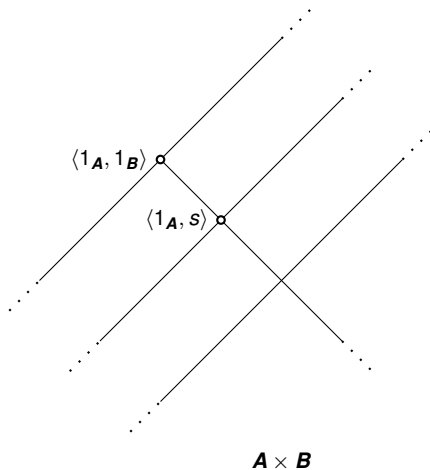
$$x \rightarrow y = \bigvee \{z \in B \mid xz \leq y\}$$

$$0_{\mathbf{B}} = 1 \text{ or } a \text{ (depending whether } \models_{\mathbf{A}} 1 \leq 0 \text{ or not).}$$

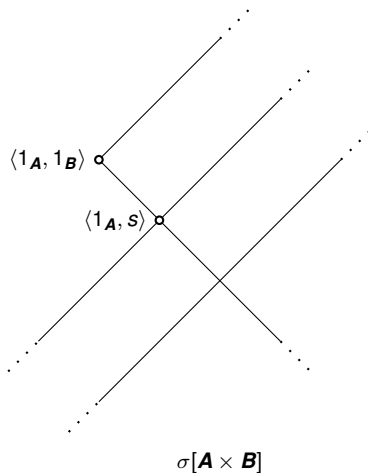
Construction of a well-connected algebra



Construction of a well-connected algebra



Construction of a well-connected algebra



Construction of a well-connected algebra

- $\mathbf{A} \times \mathbf{B}$ belongs to \mathbf{K} .
- $\sigma[\mathbf{A} \times \mathbf{B}]$ forms a subalgebra of $\mathbf{A} \times \mathbf{B}$ with respect to the language $\{\cdot, \wedge, \vee, 1, 0\}$
- $\sigma[\mathbf{A} \times \mathbf{B}]$ is an image of an interior operator σ .
- Moreover, we have $\sigma(x)\sigma(y) \leq \sigma(xy)$, i.e., σ is a conucleus.
- Thus $\sigma[\mathbf{A} \times \mathbf{B}]$ is an FL-algebra ($x \setminus_{\sigma} y = \sigma(x \setminus y)$).
- $\sigma[\mathbf{A} \times \mathbf{B}]$ is well-connected.
- \mathbf{A} is a homomorphic image of $\sigma[\mathbf{A} \times \mathbf{B}]$.

DP for ℓ -monoidal extensions

Theorem

Every quasivariety of FL-algebras defined by ℓ -monoidal quasi-identities has the DP.

*Every extension of **FL** by ℓ -monoidal rules has the DP.*

Example

- Every extension of **FL** by structural rules (e), (c), (i), (o) enjoys the DP.
- The extension of **FL** by the rule

$$\frac{\Rightarrow \varphi \cdot \psi}{\Rightarrow \varphi}$$

has the DP. It defines a proper subquasivariety of FL

\mathcal{M}_2 -axioms

- Note that $xy/y = x = y \setminus yx$ are equivalent to $xz = yz \Rightarrow x = y$ and $zx = zy \Rightarrow x = y$.

Definition (Class \mathcal{M}_2)

Let \mathcal{V} be a set of variables. Given a set T of terms, let T° be its closure under the operations $\{\cdot, \wedge, \vee, 1\}$. Likewise, let T^\bullet be its closure under the following rules:

- $0 \in T^\bullet, \mathcal{V}^\circ \subseteq T^\bullet$;
- if $t, u \in T^\bullet$ then $t \wedge u \in T^\bullet$;
- if $t \in T^\circ$ and $u \in T^\bullet$, then $t \setminus u, u/t \in T^\bullet$.

We define $\mathcal{M}_1 = \mathcal{V}^\bullet$ and $\mathcal{M}_2 = \mathcal{M}_1^\bullet$. An identity $t \leq u$ belongs to \mathcal{M}_2 if $t \in \mathcal{M}_1^\circ$ and $u \in \mathcal{M}_2$. Analogously, $\alpha \Rightarrow \beta \in \mathcal{M}_2$ if $\alpha \in \mathcal{M}_1^\circ$ and $\beta \in \mathcal{M}_2$.

Examples of \mathcal{M}_2 -axioms

Axiom	Name
$\alpha\beta \Rightarrow \beta\alpha$	exchange (e)
$\alpha \Rightarrow \mathbf{1}$	integrality, left weakening (i)
$\mathbf{0} \Rightarrow \alpha$	right weakening (o)
$\alpha \Rightarrow \alpha\alpha$	contraction (c)
$\alpha^n \Rightarrow \alpha^m$	knotted axioms ($n, m \geq 0$)
$\alpha \wedge (\alpha \setminus \mathbf{0}) \Rightarrow$	no-contradiction
$\alpha\beta/\beta \Rightarrow \alpha, \alpha \setminus \alpha\beta \Rightarrow \beta$	cancellativity
$\alpha \wedge (\beta \vee \gamma) \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$	distributivity
$((\alpha \wedge \beta) \vee \gamma) \wedge \beta \Rightarrow (\alpha \wedge \beta) \vee (\gamma \wedge \beta)$	modularity
$\alpha\beta \wedge \alpha\gamma \Rightarrow \alpha(\beta \wedge \gamma)$	(\cdot, \wedge) -distributivity
$\alpha \wedge (\beta\gamma) \Rightarrow (\alpha \wedge \beta)(\alpha \wedge \gamma)$	(\wedge, \cdot) -distributivity

DP for extensions by \mathcal{M}_2 -axioms

Theorem

Every identity in \mathcal{M}_2 is equivalent in FL to a set of ℓ -monoidal quasi-identities.

Corollary

*Every extension of **FL** by \mathcal{M}_2 -axioms has the DP.*

Involutive substructural logics

- Negations: $\sim\varphi = \varphi \setminus 0$, $-\varphi = 0/\varphi$.
- Double negation elimination laws (DN): $\sim-\varphi \Rightarrow \varphi$, $-\sim\varphi \Rightarrow \varphi$.
- Let \mathbf{L} be a substructural logic. Then \mathbf{InL} is $\mathbf{L}+(\text{DN})$.

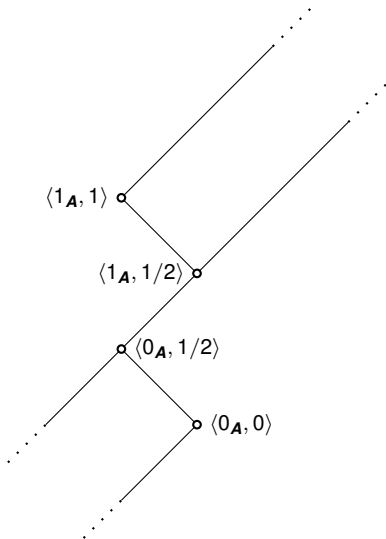
Theorem

Every extension of \mathbf{InFL} and \mathbf{InFL}_e (**MALL**) by inference rules in the language $\{\wedge, \vee, 1\}$ has the DP.

Example

The distributive extension of \mathbf{InFL}_e has the DP. Thus the relevance logic **RW** has the DP.

Construction for involutive logics



Conclusions

Theorem

Let \mathbf{L} be a consistent substructural logic. The decision problem for \mathbf{L} is coNP -hard. If \mathbf{L} further satisfies the DP, then it is PSPACE -hard.

Corollary

Let \mathbf{L} be a consistent extension of \mathbf{FL} by ℓ -monoidal inference rules and/or \mathcal{M}_2 -axioms. Then the decision problem for \mathbf{L} is PSPACE -hard. The same is true also for every consistent extension of \mathbf{InFL} or \mathbf{InFL}_e by inference rules in the language $\{\wedge, \vee, 1\}$.

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Let \mathbf{L} be a consistent substructural logic. The decision problem for \mathbf{L} is coNP-hard. If \mathbf{L} further satisfies the DP, then it is PSPACE-hard.

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The DP is a sufficient condition for PSPACE-hardness but not a necessary one. A counterexample is \mathbf{LQ} obtained by extending intuitionistic logic with the law $\neg\alpha \vee \neg\neg\alpha$.