Expansion Nets

Proof Nets for Classical Logic

Richard McKinley

University of Bern

ALCOP 2011

If you enjoyed your visit to Bern...

Why not come to “Gentzen Systems and Beyond ’11”?

Satellite workshop of TABLEAUX 2011, 4th July.

Bureaucracy in syntax

To write down a proof in the sequent calculus, we have to make arbitrary choices

$$\dfrac{\dfrac{\Gamma, A, B, C, D}{\Gamma, A \vee B, C, D}\vee}{\Gamma, A \vee B, C \vee D}\vee \qquad \text{vs} \qquad \dfrac{\dfrac{\Gamma, A, B, C, D}{\Gamma, A, B, C \vee D}\vee}{\Gamma, A \vee B, C \vee D}\vee$$

We would like a representation of proofs where such choices are not necessary.

Bureaucracy in syntax

To write down a proof in the sequent calculus, we have to make arbitrary choices

$$\dfrac{\dfrac{\Gamma, A, B, C, D}{\Gamma, A \vee B, C, D}\vee}{\Gamma, A \vee B, C \vee D}\vee \qquad \text{vs} \qquad \dfrac{\dfrac{\Gamma, A, B, C, D}{\Gamma, A, B, C \vee D}\vee}{\Gamma, A \vee B, C \vee D}\vee$$

We would like a representation of proofs where such choices are not necessary.

Abstract proof objects

We look for objects which :

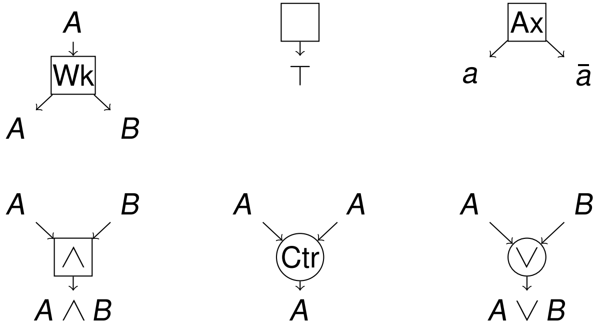
- Represent equivalence classes of sequent proofs
- under natural notions of identity of proofs
- such that proof-checking takes at worst polynomial time.
- with syntactic cut-elimination

Proof nets

Girard's proof nets [87] provide just such a framework for linear logic.

- Graph-based representation of proofs
- Inductive translation from sequent proofs to nets...
- identifying proofs differing by commuting conversions
- Correctness is polynomial time.

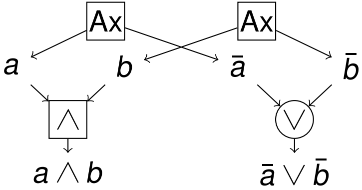
Robinson's proof nets for classical propositional logic[00]



A proof structure is a graph built from the above elements, with no incoming edges.

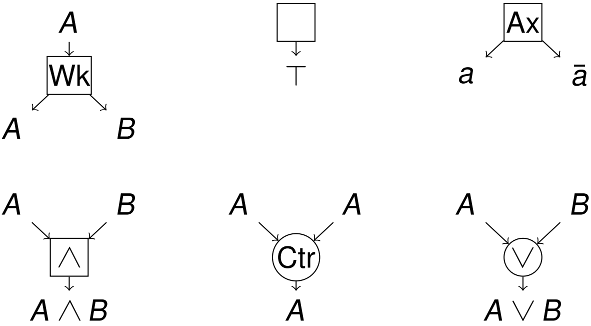
Robinson's proof nets for classical propositional logic[00]

Example of a proof structure:



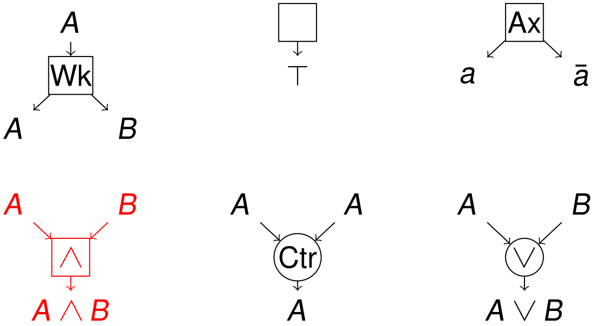
The conclusion of a proof-structure is a sequent: here it's $a \wedge b, \bar{a} \vee \bar{b}$

Robinson's proof nets for classical propositional logic[00]



Each element above corresponds to a rule of the one-sided sequent calculus

Robinson's proof nets for classical propositional logic[00]

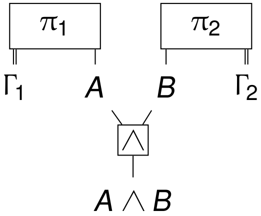


Each element above corresponds to a rule of the one-sided sequent calculus

Robinson's proof nets for classical propositional logic[00]

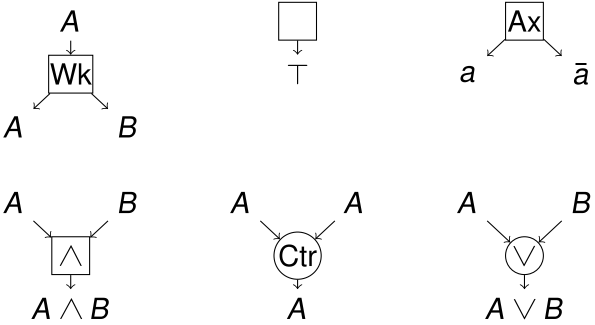
There is a translation from sequent proofs to structures

$$\frac{\vdash \Gamma_1, A \qquad \vdash \Gamma_2, B}{\vdash \Gamma_1, \Gamma_2, A \wedge B} \quad \to$$



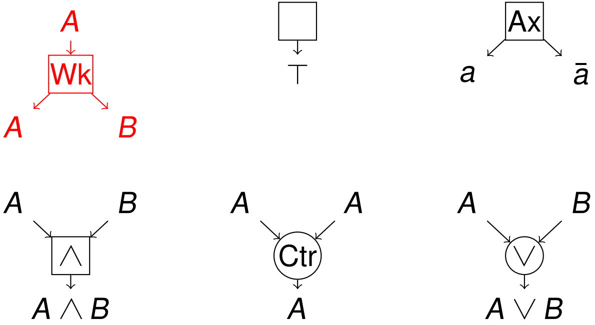
Call the translation of a sequent proof a net.

Robinson's proof nets for classical propositional logic[00]



Weakening creates problems with the translation

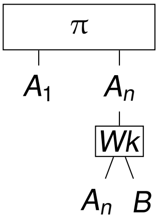
Robinson’s proof nets for classical propositional logic[00]



Weakening creates problems with the translation

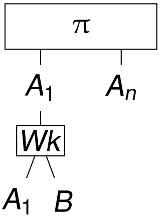
Robinson's proof nets for classical propositional logic[00]

$$\frac{\vdash A_1, \ldots A_n}{\vdash A_1, \ldots A_n, B} \quad \rightarrow$$



Robinson's proof nets for classical propositional logic[00]

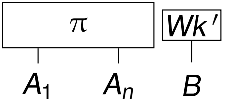
$$\frac{\vdash A_1, \ldots A_n}{\vdash A_1, \ldots A_n, B} \quad \rightarrow$$



Unattached weakening

Why not define weakening like this instead?

$$\frac{\vdash A_1, \ldots A_n}{\vdash A_1, \ldots A_n, B} \quad \rightarrow$$

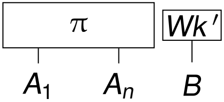


Causes problems with *correctness*

Unattached weakening

Why not define weakening like this instead?

$$\frac{\vdash A_1, \ldots A_n}{\vdash A_1, \ldots A_n, B} \quad \to$$



Causes problems with *correctness*

The problem with weakening

- With Robinson's weakening,
  - Decide if a structure comes from a sequent proof in polynomial time
  - No canonical map from proofs to structures
- Without weakening attachment
  - Canonical map from sequent proofs to stuctures
  - Correctness is NP-complete.

The problem with weakening

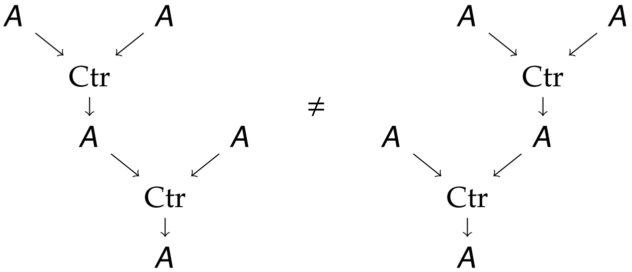
- With Robinson's weakening,
  - Decide if a structure comes from a sequent proof in polynomial time
  - No canonical map from proofs to structures
- Without weakening attachment
  - Canonical map from sequent proofs to stuctures
  - Correctness is NP-complete.

Proposed solutions

- Lamarche-Strassburger[05]: $\mathbb{B}/\mathbb{N}$-nets.
- Hughes [06]: Combinatorial proofs.

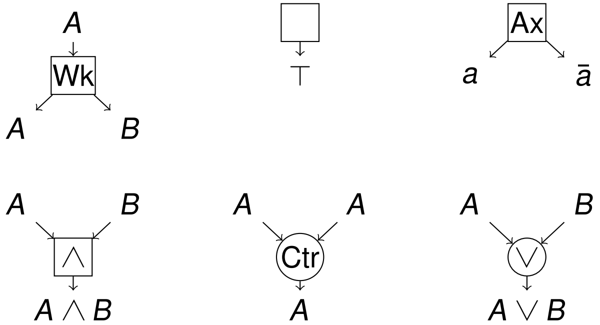
Both approaches fail to capture equivalence classes of sequent proofs.

The (smaller) problem with binary contraction



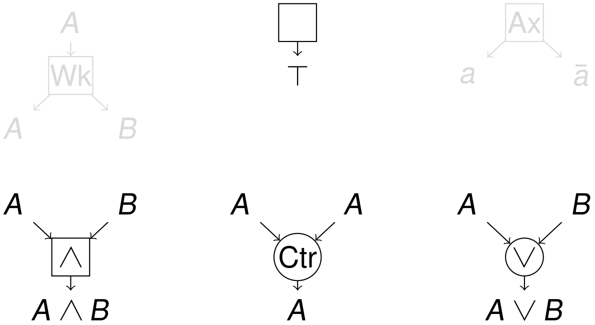
and other problems concerning the interaction between contractions and weakenings, or contractions and disjunctions.

From graphs to linked forests



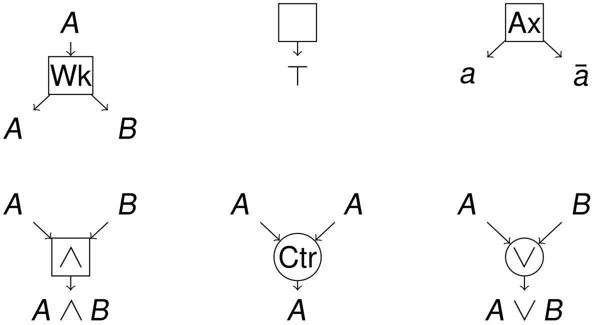
A proof net as a multiset F of typed trees with a set of “links”.

From graphs to linked forests



A proof net as a multiset F of typed trees with a set of "links".

From graphs to linked forests



A proof net as a multiset F of typed trees with a set of “links”.

The e-annotated sequent calculus $\mathbf{LK}_{ed}$

$$\frac{}{\vdash 1 : \top}\ \mathrm{Ax}_\top \qquad \frac{}{(\bar{x}) : \bar{p},\ (x) : p}\ \mathrm{Ax}$$

$$\frac{F,\ t : A,\ s : B}{F,\ t \vee s : A \vee B}\ \vee \qquad \frac{F,\ t : A \qquad G,\ s : B}{F,\ (t \otimes s) : A \wedge B}\ \wedge$$

$$\frac{F}{F,\ * : A}\ \mathrm{W} \qquad \frac{F \quad G}{F,\ G}\ \mathrm{Mix}$$

$$\frac{F,\ t : A \wedge B,\ s : A \wedge B}{F,\ t + s : A \wedge B}\ \mathrm{C}_\wedge \qquad \frac{F,\ s : p,\ t : p}{F,\ s + t : p}\ \mathrm{C}_p \qquad \frac{F,\ s : \bar{p},\ t : \bar{p}}{F,\ s + t : \bar{p}}\ \mathrm{C}_{\bar{p}}$$

The e-annotated sequent calculus $\mathbf{LK}_{ed}$

$$\frac{}{\vdash 1 : \top}\,\mathrm{Ax}_\top \qquad \frac{}{(\bar{x}) : \bar{p},\ (x) : p}\,\mathrm{Ax}$$

$$\frac{F,\ t : A,\ s : B}{F,\ t \vee s : A \vee B}\,\vee \qquad \frac{F,\ t : A \qquad G,\ s : B}{F,\ (t \otimes s) : A \wedge B}\,\wedge$$

$$\frac{F}{F,\ * : A}\,\mathrm{W} \qquad \frac{F \quad G}{F,\ G}\,\mathrm{Mix}$$

$$\frac{F,\ t : A \wedge B,\ s : A \wedge B}{F,\ t + s : A \wedge B}\,\mathrm{C}_\wedge \qquad \frac{F,\ s : p,\ t : p}{F,\ s + t : p}\,\mathrm{C}_p \qquad \frac{F,\ s : \bar{p},\ t : \bar{p}}{F,\ s + t : \bar{p}}\,\mathrm{C}_{\bar{p}}$$

The e-annotated sequent calculus $\mathbf{LK}_{ed}$

$$\frac{}{\vdash 1 : \top}\ \mathrm{Ax}_\top \qquad \frac{}{(\bar{x}) : \bar{p},\ (x) : p}\ \mathrm{Ax}$$

$$\frac{F,\ t : A,\ s : B}{F,\ t \vee s : A \vee B}\ \vee \qquad \frac{F,\ t : A \qquad G,\ s : B}{F,\ (t \otimes s) : A \wedge B}\ \wedge$$

$$\frac{F}{F,\ * : A}\ \mathrm{W} \qquad \frac{F \quad G}{F,\ G}\ \mathrm{Mix}$$

$$\frac{F,\ t : A \wedge B,\ s : A \wedge B}{F,\ t + s : A \wedge B}\ \mathrm{C}_\wedge \qquad \frac{F,\ s : p,\ t : p}{F,\ s + t : p}\ \mathrm{C}_p \qquad \frac{F,\ s : \bar{p},\ t : \bar{p}}{F,\ s + t : \bar{p}}\ \mathrm{C}_{\bar{p}}$$

The e-annotated sequent calculus $\mathbf{LK}_{ed}$

$$\frac{}{\vdash 1 : \top}\,\mathrm{Ax}_\top \qquad \frac{}{(\bar{x}) : \bar{p},\ (x) : p}\,\mathrm{Ax}$$

$$\frac{F,\ t : A,\ s : B}{F,\ t \vee s : A \vee B}\,\vee \qquad \frac{F,\ t : A \qquad G,\ s : B}{F,\ (t \otimes s) : A \wedge B}\,\wedge$$

$$\frac{F}{F,\ * : A}\,\mathrm{W} \qquad \frac{F \quad G}{F,\ G}\,\mathrm{Mix}$$

$$\frac{F,\ t : A \wedge B,\ s : A \wedge B}{F,\ t + s : A \wedge B}\,\mathrm{C}_\wedge \qquad \frac{F,\ s : p,\ t : p}{F,\ s + t : p}\,\mathrm{C}_p \qquad \frac{F,\ s : \bar{p},\ t : \bar{p}}{F,\ s + t : \bar{p}}\,\mathrm{C}_{\bar{p}}$$

The e-annotated sequent calculus $\mathbf{LK}_{ed}$

$$\dfrac{}{\vdash 1 : \top}\ \mathrm{Ax}_{\top} \qquad \dfrac{}{(\bar{x}) : \bar{p},\ (x) : p}\ \mathrm{Ax}$$

$$\dfrac{F,\ t : A,\ s : B}{F,\ t \vee s : A \vee B}\ \vee \qquad \dfrac{F,\ t : A \qquad G,\ s : B}{F,\ (t \otimes s) : A \wedge B}\ \wedge$$

$$\dfrac{F}{F,\ * : A}\ \mathrm{W} \qquad \dfrac{F \quad G}{F,\ G}\ \mathrm{Mix}$$

$$\dfrac{F,\ t : A \wedge B,\ s : A \wedge B}{F,\ t + s : A \wedge B}\ \mathrm{C}_{\wedge} \qquad \dfrac{F,\ s : p,\ t : p}{F,\ s + t : p}\ \mathrm{C}_{p} \qquad \dfrac{F,\ s : \bar{p},\ t : \bar{p}}{F,\ s + t : \bar{p}}\ \mathrm{C}_{\bar{p}}$$

The e-annotated sequent calculus $\mathbf{LK}_{ed}$

$$\frac{}{\vdash 1 : \top}\,\mathrm{Ax}_\top \qquad \frac{}{(\bar{x}) : \bar{p},\ (x) : p}\,\mathrm{Ax}$$

$$\frac{F,\ t : A,\ s : B}{F,\ t \vee s : A \vee B}\,\vee \qquad \frac{F,\ t : A \qquad G,\ s : B}{F,\ (t \otimes s) : A \wedge B}\,\wedge$$

$$\frac{F}{F,\ * : A}\,\mathrm{W} \qquad \frac{F \quad G}{F,\ G}\,\mathrm{Mix}$$

$$\frac{F,\ t : A \wedge B,\ s : A \wedge B}{F,\ t + s : A \wedge B}\,\mathrm{C}_\wedge \qquad \frac{F,\ s : p,\ t : p}{F,\ s + t : p}\,\mathrm{C}_p \qquad \frac{F,\ s : \bar{p},\ t : \bar{p}}{F,\ s + t : \bar{p}}\,\mathrm{C}_{\bar{p}}$$

The e-annotated sequent calculus $\mathbf{LK}_{ed}$

$$\dfrac{}{\vdash 1 : \top}\ \mathrm{Ax}_\top \qquad \dfrac{}{(\bar{x}) : \bar{p},\ (x) : p}\ \mathrm{Ax}$$

$$\dfrac{F,\ t : A,\ s : B}{F,\ t \vee s : A \vee B}\ \vee \qquad \dfrac{F,\ t : A \qquad G,\ s : B}{F,\ (t \otimes s) : A \wedge B}\ \wedge$$

$$\dfrac{F}{F,\ * : A}\ \mathrm{W} \qquad \dfrac{F \quad G}{F,\ G}\ \mathrm{Mix}$$

$$\dfrac{F,\ t : A \wedge B,\ s : A \wedge B}{F,\ t + s : A \wedge B}\ \mathrm{C}_\wedge \qquad \dfrac{F,\ s : p,\ t : p}{F,\ s + t : p}\ \mathrm{C}_p \qquad \dfrac{F,\ s : \bar{p},\ t : \bar{p}}{F,\ s + t : \bar{p}}\ \mathrm{C}_{\bar{p}}$$

The e-annotated sequent calculus $\mathbf{LK}_{ed}$

$$\dfrac{}{\vdash 1 : \top}\,\mathrm{Ax}_\top \qquad \dfrac{}{(\bar{x}) : \bar{p},\ (x) : p}\,\mathrm{Ax}$$

$$\dfrac{F,\ t : A,\ s : B}{F,\ t \vee s : A \vee B}\,\vee \qquad \dfrac{F,\ t : A \qquad G,\ s : B}{F,\ (t \otimes s) : A \wedge B}\,\wedge$$

$$\dfrac{F}{F,\ * : A}\,\mathrm{W} \qquad \dfrac{F \quad G}{F,\ G}\,\mathrm{Mix}$$

$$\dfrac{F,\ t : A \wedge B,\ s : A \wedge B}{F,\ t + s : A \wedge B}\,\mathrm{C}_\wedge \qquad \dfrac{F,\ s : p,\ t : p}{F,\ s + t : p}\,\mathrm{C}_p \qquad \dfrac{F,\ s : \bar{p},\ t : \bar{p}}{F,\ s + t : \bar{p}}\,\mathrm{C}_{\bar{p}}$$

Completeness

Forgetting the annotating trees yields a sequent-calculus complete for propositional classical logic:

Theorem

A sequent $A_1, \ldots A_n$ of propositional logic is provable in ***LK*** *if and only if there are terms $t_1, \ldots, t_n$ with*

$$\boldsymbol{LK}_{ed} \vdash t_1 : A_1, \ldots t_n : A_n$$

Completeness

Forgetting the annotating trees yields a sequent-calculus complete for propositional classical logic:

Theorem

A sequent $A_1, \ldots A_n$ of propositional logic is provable in ***LK*** *if and only if there are terms $t_1, \ldots, t_n$ with*

$$\boldsymbol{LK}_{ed} \vdash t_1 : A_1, \ldots t_n : A_n$$

Example

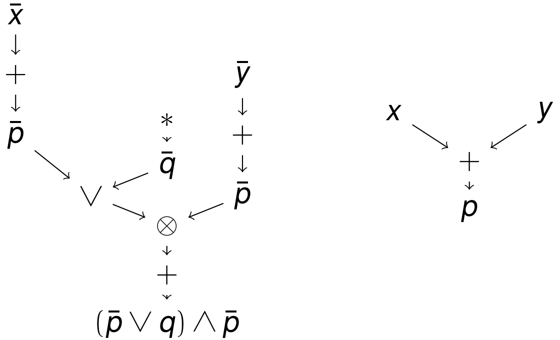
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(\bar{x}) : \bar{a},\ (x) : a}\text{Ax} \qquad \dfrac{}{(\bar{y}) : \bar{a},\ (y) : a}\text{Ax}}{(\bar{x}) : \bar{a},\ (\bar{y}) : \bar{a}, (x \otimes y) : a \wedge a}\wedge \qquad \dfrac{}{(\bar{z}) : \bar{a},\ (z) : a}\text{Ax}}{(\bar{x}) : \bar{a},\ (\bar{y}) : \bar{a},\ \bar{z} : \bar{a},\ ((x \otimes y) \otimes z) : (a \wedge a) \wedge a}\wedge}{(\bar{x}) : \bar{a},\ (\bar{y} + \bar{z}) : \bar{a},\ ((x \otimes y) \otimes z) : (a \wedge a) \wedge a}\text{C}}{(\bar{x} + \bar{y} + \bar{z}) : \bar{a},\ ((x \otimes y) \otimes z) : (a \wedge a) \wedge a}\text{C}$$

Annotated sequents and Proof nets

The following annotated sequent represents a proof of Pierce's law

$$(((\bar{x}) \vee *) \otimes (\bar{y})) : (\bar{p} \vee q) \wedge \bar{p}, \quad (x + y) : p$$

The graph of this annotated sequent is

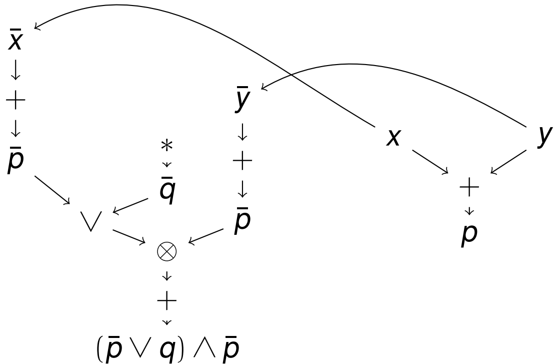


Annotated sequents and Proof nets

The following annotated sequent represents a proof of Pierce's law

$$(((\bar{x}) \vee *) \otimes (\bar{y})) : (\bar{p} \vee q) \wedge \bar{p}, \quad (x + y) : p$$

The graph of this annotated sequent is



Sequentialization

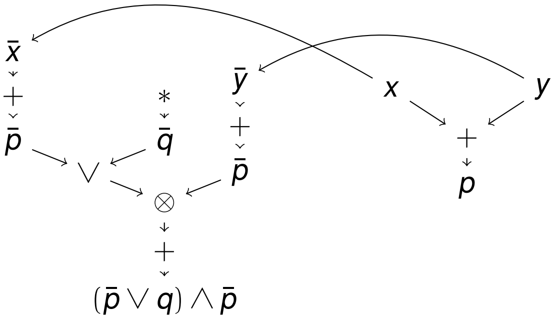
Correctness based on usual proof-net correctness techniques.

Theorem

An annotated sequent F is correct if and only if $\vdash F$ can be derived in the annotated system.

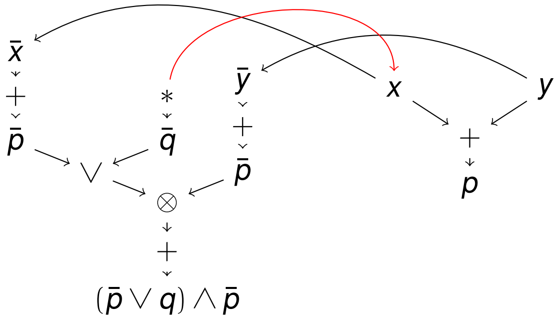
Weakening attachment

Correctness for annotated sequents is exponential-time, because we need to find an *attachment* for the weakenings:



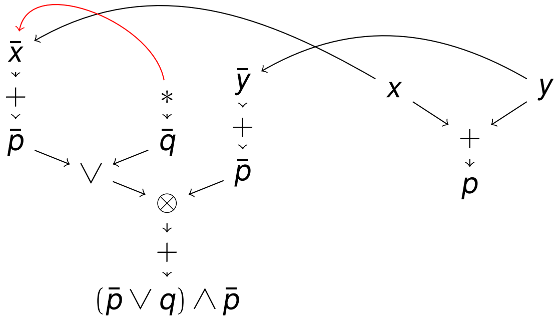
Weakening attachment

Correctness for annotated sequents is exponential-time, because we need to find an *attachment* for the weakenings:



Weakening attachment

Correctness for annotated sequents is exponential-time, because we need to find an *attachment* for the weakenings:



Default attachments

If the subtree $*$ appears in a disjunction $* \vee t$ or $t \vee *$, such that $t \neq *$, then it has a *default* attachment, namely t.

Checking correctness for forests in which every $*$ is default-attached can be done in polynomial time.

Default attachments

If the subtree $*$ appears in a disjunction $* \vee t$ or $t \vee *$, such that $t \neq *$, then it has a *default* attachment, namely t.

Checking correctness for forests in which every $*$ is default-attached can be done in polynomial time.

The sequent calculus **LK***

$$\frac{}{a,\ \bar{a}}\ \mathsf{Ax} \qquad \frac{}{\top}\ \mathsf{Ax}_\top$$

$$\frac{\Gamma,\ A}{\Gamma, A \vee B}\ \vee_0 \qquad \frac{\Gamma,\ A, B}{\Gamma, A \vee B}\ \vee \qquad \frac{\Gamma,\ B}{\Gamma, A \vee B}\ \vee_1$$

$$\frac{\Gamma,\ a, a}{\Gamma, a}\ \mathsf{C} \qquad \frac{\Gamma,\ \bar{a}, \bar{a}}{\Gamma, \bar{a}}\ \mathsf{C} \qquad \frac{\Gamma,\ A \wedge B, A \wedge B}{\Gamma, A \wedge B}\ \mathsf{C}$$

$$\frac{\Gamma, A \qquad \Delta, B}{\Gamma, \Delta,\ A \wedge B}\ \wedge \qquad \frac{\Gamma \qquad \Delta}{\Gamma, \Delta}\ \mathrm{Mix}$$

E-annotating **LK***

$$\frac{}{1 : \top}\,\mathrm{Ax}_\top \qquad \frac{}{(\bar{x}) : \bar{p},\ (x) : p}\,\mathrm{Ax}$$

$$\frac{F,\ t : A,\ s : B}{F,\ t \vee s : A \vee B}\,\vee \qquad \frac{F,\ t : A}{F,\ t \vee * : A \vee B}\,\vee_0 \qquad \frac{F,\ s : B}{F,\ * \vee s : A \vee B}\,\vee_1$$

$$\frac{F,\ t : A \wedge B,\ s : A \wedge B}{F,\ t + s : A \wedge B}\,\mathrm{C}_\wedge \qquad \frac{F,\ s : p,\ t : p}{F,\ s + t : p}\,\mathrm{C}_p \qquad \frac{F,\ s : \bar{p},\ t : \bar{p}}{F,\ s + t : \bar{p}}\,\mathrm{C}_{\bar{p}}$$

$$\frac{F \quad G}{F,\ G}\,\mathrm{Mix} \qquad \frac{F,\ t : A \qquad G,\ s : B}{F, G,\ (t \otimes s) : A \wedge B}\,\wedge$$

Expansion-nets

A is a theorem of propositional classical logic if and only if $\mathbf{LK}^* \vdash t : A$ for some t.

Given an arbitrary t, we can check if $\mathbf{LK}^* \vdash t : A$ in polynomial time.

Two derivations of $t : A$ differ by rule permutations and rearrangements of contractions.

Expansion-nets

A is a theorem of propositional classical logic if and only if $\mathbf{LK}^* \vdash t : A$ for some t.

Given an arbitrary t, we can check if $\mathbf{LK}^* \vdash t : A$ in polynomial time.

Two derivations of $t : A$ differ by rule permutations and rearrangements of contractions.

Expansion-nets

A is a theorem of propositional classical logic if and only if $\mathbf{LK}^* \vdash t : A$ for some t.

Given an arbitrary t, we can check if $\mathbf{LK}^* \vdash t : A$ in polynomial time.

Two derivations of $t : A$ differ by rule permutations and rearrangements of contractions.

Discussion of Cut-elimination (if time permits)

Cut-reduction in **LK*** is *non-local*.

Cut-reduction in classical proof-nets is *always* non-local: one deletes/duplicates subnets.

Discussion of Cut-elimination (if time permits)

Cut-reduction in $\mathbf{LK}^*$ is *non-local*.

Cut-reduction in classical proof-nets is *always* non-local: one deletes/duplicates subnets.

Conclusions, further work

Expansion nets represent equivalence classes of sequent proofs, are canonical, and have polynomial-time correctness.

Further work:

- Strong normalization/weakly normalizing subsystems
- Equivalence of proofs containing cuts
- First/Higher-order logic
- Computational interpretation (Curry-Howard)